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# THE PARAMETERIZATION METHOD FOR INVARIANT MANIFOLDS OF REAL-ANALYTIC DYNAMICAL SYSTEMS

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# Contents

# Introduction

## Aims

The main goal of this work is to devolope the parameterization method, which was introduced by X. Cabré, E. Fontich and R. de la Llave [HCF$^+$16]. It is an important tool to study diverse invariant manifolds attached to fixed points in different contexts. To be acquainted with the parameterization method, we divide the work into two studies which are as follows.

In the first study, we aim to prove the existence and regularity of invariant manifolds. Furthermore, we also demonstrate that the parameterization method in different contexts can reach to obtain different kinds of invariant manifold at fixed points. As a first simple application, the method allows us to give a quick proof of (un)stable manifolds theorems. For instance, the existence of a real-analytic one-dimensional stable manifolds at the origin for maps or a 2D stable manifolds for flows.

Once, we proved the existence of the manifolds. The second study is to emphasize the computational aspects derived from the application of the parameterization method. We would like to work out the coefficients of the invariant manifold expanded in series and sketch the approximation of the invariant manifolds by using computer programs. We can get an efficient algorithm for numerical computation of invariant manifolds base on the parameterization method.

## Structure

As we mentioned before the work is mainly divided into two parts. The first part will be sequentially explained 1, 2, 3 and 4. The second part will be described in Chapter 5. To be more precise, we will give a brief description for each chapter.

In Chapter 1 we state the parameterization method for maps and flows. In the casee of maps, we introduce first the theorem for real-analytic stable manifolds as an application. As motivation to follow, we show the technique that we use to obtain the power series expansions of parameterization of invariant manifold of maps at fixed point. Secondly, we state a general theorem of non-resonance invariant manifolds for maps. Finally, the theorem of non-resonance invariant manifolds for flows is introduced.

In Chapter 2 we recall the necessary prerequisities for the subsequent chapters. For instance, the notion of Banach spaces are used to be a infinite dimensional spaces. Because the parameterization method is solved in an appropriate Banach space. Then we introduce the continuous linear and k-linear maps , we need to pay attention to the notion of continuity, because it is different than usual since we are in infinite dimensional spaces. This leads to the concept of operator norm. Moreover, the space $S^K(E;F)$ forms by homogeneous polynomial of degree k from E to F will be defined and will play an essential role in the parameterization method. We have introduced the continuity before, the next step is to define the differentiability in Banach spaces where the differential of $f$ is a multilinear map and some properties will be found familiar. Since the spaces we will deal with are Banach spaces, thus we state the Taylor's formula, the converse Taylor's Theorem using the Landau's $o$-notation and the Implicit Function Theorem for the Banach spaces.

After recalling the prerequisities, in Chapter 3 we focus on the Banach spaces of analytic functions which plays an important role in the parameterization method. Firstly we define the space of analytic functions in Banach spaces $\mathcal{A}_\delta(E;F)$. After that, we state some Lemmas and Propositions about the upper bounded norm of $f \in \mathcal{A}_\delta(E;F)$ or the $i$-th derivate of $f$. Take everthing into account, we state and prove the so-called Omega-Lemma that guarantees the operator of the composition of analytic functions is a $C^\infty$ operator( in fact, is analytic operator).

Once we have prepared all the ingredients, in Chapter 4 we prove the parameterization in different contexts. Even though the contexts is different, the strategy are similar to maps or flows.

In Chapter 5, we present two numerical applications. The first one is Hénon map, applying the parameterization method we are able to prove the existence of 1D stable and unstable manifold for maps. In addition, we did a computational algorithm so as to draw the (un)stable manifolds and the attractor. The second is the so-called Lorenz system, for which we compute 1D unstable manifolds and 2D stable manifolds of th origin. In both cases, we can observe a remarkable aspect: the closure of unstable manifolds contains the strange attractors.

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# Chapter 1

# The parameterization method

As we mentioned we aim to present the parameterization method for invariant manifolds of real-analytic dynamic systems. It is an important tool in the theory of invariant manifold due to establish the existence and regularity of invariant manifolds attached to fixed point. In addition, it is applicable to differents type of manifolds in different contexts for instance analytic manifolds, finitely differentiable manifolds. In our case we deal with the analytic mainfolds, but we emphasize it can be applied to many others cases.

## 1.1 The parameterization method for maps

We consider $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F$ has a fixed point at the origin F(0) = 0 throughout this chapter. We want to study the aspects of the dynamics in a neighborhood of the fixed point. The heuristics idea is that in a small neighborhood, the map is very similar to its linear part A:=DF(0). If there are the subspaces $E \subset \mathbb{R}^d$ invariant under the linearization A, one can hope that there exist smooth manifolds tangent to E at the origin which are invariant under the map $F$.

Rougly Speaking, the method for maps consists in looking for a parameterization $K : U_1 \subset \mathbb{R}^d \longrightarrow U$ of the invariant manifold and $R : U_1 \longrightarrow U_1$ is a representation of the dynamics of the map F restricted to the manifold in such way to hold

$$F(K(s)) = K(R(s)) \quad \text{for } s \in U_1 \tag{1.1}$$

The preceding equation guarantees that $K(U_1)$ is an invariant manifold of F. In the following sections we will show differents non-resonance conditions depend on the contexts. In addition, we will demonstrate the above equation can be solved provided that the non-resonance conditions are hold.

The method for flows will be illustrated in following section where the equation to be solved is different. Even the equation is different the leading idea is similar to case for maps.

The strategy to prove the method in both case is similar. We write $K(s) = \sum_{m=0}^{\infty} K_m s^m$, $R(s) = \sum_{m=0}^{\infty} R_m s^m$ and solve 1.1 by matching the powers, we get the term $K_m$ and also R. The next step is formulate K as $K = K^{\leq} + K^{>}$ where $K^{\leq}$ is a polynomial of degree L and $K^{>}$ vanishes at the origin with its first L derivatives. Finally, we will write a functional equation for $K^{>}$ which will be solved by applying Implicit Function theorem in an appropriate Banach space.

Let us focus on the case $d = 1$ as a motivation to understand the parameterization method and later we generalize for $d > 1$. Suppose there exists $\lambda \in Spec(A)$ where $Spec(A)$ stands for the spectrum of the linear operator $A := DF(0)$ and $|\lambda| < 1$. Let E be a subspace generated by an eigenvector of $\lambda$ such that is invariant to the linear part.

We want to find a 1D invariant manifold $\mathcal{K}$ such that is tangent to E at 0, that is, look for a map $K : U_1 \subset \mathbb{R} \to \mathbb{R}^n$ that parameterizes $\mathcal{K} := K(U_1)$ and a map $R : U_1 \to U_1$ as simple as possible for instance $R(s) = \lambda s$ such that satisfy

$$F(K(s)) = K(R(s)) \tag{1.2}$$

Using Talyor expansion K that equals to

$$K(s) = \sum_{m=1}^{\infty} K_m s^m, \quad \text{where} \quad K_m = \frac{1}{m!}\frac{d^m}{ds^m}K_m(0) \in \mathbb{R}^n$$

For m = 1, $K$ must hold $DF(0)K_m - \lambda K_m = 0$. Hence, $AK_1 = \lambda K_1$, i.e,$K_1$ is an eigenvector of eigenvalues $\lambda$. For m > 1, we could get the term $K_m$ in recursive way. Assuming $K_1, K_2 \cdots K_{m-1}$ are known and try to obtain $K_m$ depending on $\{K_1, \cdots, K_{m-1}\}$(an derivatives of F up to order m. As F is an analytic map, we could express as a power series in neighborhood of 0 in $\mathbb{C}^n$.).

We rewrite (1.2) and only care about the terms up to order m:

$$\begin{aligned} F(K(s)) - K(\lambda s) &= F(K_{<m}(s) + K_m s^m + \cdots) - K_{<m}(\lambda s) - K_m \lambda^m s^m - \cdots \\ &\overset{(a)}{=} F(K_{<m}(s)) + DF(K_{<m}(s))K_m s^m + \frac{1}{2}D^2F(K_{<m}(s))(K_m s^m)^{\otimes 2} + \cdots \\ &\quad - K_{<m}(\lambda s) - K_m \lambda^m s^m - \cdots \\ &\overset{(b)}{=} [F(K_{<m})(s)]_{\leq m} + DF(0)K_m s^m - K_{<m}(\lambda s) - K_m \lambda^m s^m + \cdots = 0, \end{aligned} \tag{1.3}$$

where $K_{<m}(s)$ indicates the terms up to order m-1 of K(s), $[F(K_{<m})(s)]_{\leq m}$ indicates the terms up to order m of $F(K_{<m})(s)$. In the second equality we expand the Taylor series of F in $K_{<m}(s)$ and the last one we expand $DF(K_{<m}(s))K_m s^m = DF(0)K_m s^m + \cdots$ and since $\frac{1}{2}DF^2(K_{<m}(s))(K_m s^m)^{\otimes 2}$ is the higher order than order m, we do not care about it.

If we try to solve (1.3) by equating the m-th term of s to 0

$$AK_m s^m - K_m \lambda^m s^m = -E_m s^m$$

where $E_m = [F(K_{<m})(s) - K_{<}(\lambda s)]_m = [F(K_{<m})(s)]_m$.
Provided $\forall m \geq 2 \;\; \lambda^m \notin Spec(A)$, we get

$$K_m = -(A - \lambda^m I)^{-1}[F(K_{<m})(s)]_m \quad \forall m \geq 2$$

We emphasize that the method to obtain $K^{\leq m}$ is algebraic in nature.

After obtaining the coefficients of the power series $K(s) = \sum_{m\leq 1} K_m s^m$. The question is proving the convergence, that is, $K$ is a real-analytic(as long as F is real-analytic).

There are several ways to do so. The classical one is using the so-called majorant method, based on finding recurrently bounds by the norms of the coefficients. Our approach follows a more fucntional perspective, that can be generalized to finitely smooth and $C^\infty$ contexts.

The functional approach consider the functional equation

$$\mathcal{F}(K, R) := F \circ K - K \circ R \tag{1.4}$$

where the unkowns are $K$ and $R$. $R$ will be a polynomial depending on the finitely order terms of $F$ ans $Spec(A)$. Then, we can write (1.2) as $\mathcal{F}(K,R) = 0$. Later in the Chapter 4 we can see there is an appropiate space which is a Banach space such that $\mathcal{F}(K,R) = 0$.

Note that in (1.4) appear the composition of functions in Banach space. The operator of composition of analytic functions is well defined? Moreover, is it analytic or $C^\infty$? Hence, the notion of Banach space, differentiabiblity in Banach spaces,... are needed and will appear in Chapter 2 and 3.

### 1.1.1 Analytic one-dimensional stable manifolds

In this subsection we state a theorem for the parameterization method in case of 1D stable manifolds. Its proof is an excellent motivation to understand the parameterization method and is also as an example to illustrate the leading appeared of the proof of the method in the higher dimensional. It can be found in Chapter 4.

**Theorem 1.1.** *Let $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a real-analytic map in a neighborhood $U$ of 0, with $F(0) = 0 \in \mathbb{R}^n$ and $A := DF(0)$. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A$ and let $v \in \mathbb{R}^d \backslash \{0\}$ sastify $Av = \lambda v$. Let us assumme:*

*(i)* *$A$ is invertible;*

*(ii)* $0 < |\lambda| < 1$;

*(iii)* $\lambda^j \notin Spec(A) \quad \forall j \geq 2$ *(non-resonance condition).*

*Then, there exists an real-analytic map $K : U_1 \subset \mathbb{R} \longrightarrow \mathbb{R}^n$, where $U_1$ is an star shaped open neighbourhood of 0 in $\mathbb{R}$, sastifying*

$$F(K(x)) = K(\lambda x) \quad \text{in} \;\; U_1 \tag{1.5}$$

*$K(0) = 0$ , and $K'(0) = v$. Hence, the image of $K$ is an real-analytic one-dimensional manifold invariant under $F$ and tangent to $v$ at the origin.*
*Moreover, the dynamics on the invariant manifold is conjugated to the linear map $x \mapsto \lambda x$ in the space of parameters.*
*In addition, if $\widehat{K}$ is real-another analytic solution of $F \circ K = K \circ \lambda$ in a neighborhood of the origin, with $\widehat{K}(0) = 0$ and $\widehat{K}'(0) = \beta v$ for some $\beta \in \mathbb{R}$ , then $\widehat{K}(t) = K(\beta t)$ for $t$ small enough.*

**Remark 1.2.** Even though it is stated for the stable manifold,this theorem is hold for unstable manifold using the theorem to invers of $F$. Since $F$ is local biholomorphism (local invers $F^{-1}$ that is holomorphic).

On other hand, as we mentioned before. The parameterization method is an efficient algorithm for the numerical computation. In Chapter 4, we compute the stable and unstable maninfols of Hénon map attached to fixed point. Furthermore, we compute the Hénon attractor in order to compare with the unstable manifold.

### 1.1.2 Non-resonant invariant manifolds for maps

The heuristic idea is that, given a real-analytic map $F$ in $\mathbb{R}^n$ such that $F(0) = 0$ and $E \subset \mathbb{R}^d$ is the linear subespace invariant by the linearization. Satisfying non-resonance conditions there exists a real-analytic map $K : U_1\mathbb{R}^d \to \mathbb{R}^n$ and the map R such that $F \circ K - K \circ R = 0$ where $K(0) = 0$ and $R(0) = 0$. These condition will ensure that is an invariant manifold $\mathcal{K} = K(U_1)$ through the origin. Suppose $(K, R)$ is a solution of $F \circ K - K \circ R = 0$ and we differentiate it.

$$DF(K(s))DK(s) = DK(R(s))DR(s)$$

evaluate at $s = 0$

$$DF(0)DK(0) = DK(0)DR(0) \tag{1.6}$$

where 1.6 indicates that the tangent space of the invariant manifold at 0 is invariant by the linearization. In addition, this linear subspace is a linear approximation of the invariant manifold.

To state the theorem, we first recall some standard terminology.

$$Spec(A)^j := \{\lambda_1 \cdot \lambda_2 \cdots \lambda j \mid \lambda_i \in Spec(A)\}$$

The leading idea is to prove the theorem 1.3 that is similar to case 1D. Pay attetion $s \in \mathbb{R}^d$ for $d \geq 1$ and here $K_i(s)$ is a continuous multilinear maps evaluate a $(s, \overset{i}{\cdots}, s)$. In fact, it is homogenous polynomial of degree $i$. Therefore, to prove this theorem we have to understand some notion such as continuous multilinear maps will be introuduced in the chapter 2 and its proof is in Chapter 4.

**Theorem 1.3.** *Let F: U$\subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a real-analytic map in a neighborhood U of $0 \in \mathbb{R}^n$, with F(0) = 0 and A:= DF(0). Suppose there is L $\in \mathbb{N}$ and exist a linear subspace E of $\mathbb{R}^n$ such that A(E) $\subset$ E. Hence there is a descomposition $\mathbb{R}^n = E \oplus C$, then A has the form*

$$A = \begin{bmatrix} A_E & B \\ 0 & A_C \end{bmatrix}$$

*such that:*

*(i) A is invertible.*

*(ii) $Spec(A_E) \subset \{z \in \mathbb{C} \mid |z| < 1\}$.*

*(iii) $Spec(A_E)^j \cap Spec(A_C) = \emptyset$ for j = 2,...,L. (non-resonance condition)*

*(iv) $(Spec(A_E))^{L+1}Spec(A^{-1}) \subset \{z \in \mathbb{C} \mid |z| < 1\}$.(non-resonance condition)*

*Then, there exists a real-analytic map $K : U_1 \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$, where $U_1$ is an open neighborhood of 0 $\in \mathbb{R}^d$, and a polynomial $R : U_1 \longrightarrow U_1$ of degree at most L, such that*

$$F \circ K = K \circ R \ \ in \ \ U_1, \tag{1.7}$$

$$K(0) = 0, \quad DK(0)\mathbb{R}^d = E \quad where\ DK(0) : \mathbb{R}^d \to \mathbb{R}^n, \tag{1.8}$$

$$R(0) = 0, \quad DR(0) = A_E. \tag{1.9}$$

**Remark 1.4.** $DK(0)\mathbb{R}^d = E$ in (1.8), that is, the column vectors of $DK(0)$ that are the partial derivatives expand to the linear space E.

## 1.2 The parameterization method for flows

The same idea used for maps is translated to study the invariant manifolds for differential equations. Given a vector field $\mathcal{X}$ in $U \subset \mathbb{R}^n$ with $\mathcal{X}(0) = 0$. Suppose there is a subspace $E \subset \mathbb{R}^d$ invariant by its linearization $D\mathcal{X}(0)$. The heuristic idea is that exist a real-analytic map $K : U_1 \subset \mathbb{R}^d \to \mathbb{R}^n$ such that parameterize the manifold $\mathcal{K}$ and $R : U_1 \to U_1$ is a representation of the dynamic of the vector field $\mathcal{X}$ restricted to $\mathcal{K}$ satisfying

$$F(K(s)) = DK(s) \cdot R(s) \quad \text{for } \forall s \in U_1$$

Note that the equation above ensures that the image $\mathcal{K}$ of K is invariant by the flow of $\mathcal{X}$.

$$jSpec(A) := \{\lambda_1 + \lambda_2 + \cdots + \lambda j \mid \lambda_i \in Spec(A)\}$$

**Theorem 1.5.** *Let $\mathcal{X}$ be a real-analytic vector field on an open set $U$ of $\mathbb{R}^n$ with $0 \in U$, such that $\mathcal{X}(0) = 0$. Let $A = D\mathcal{X}(0)$ and $L \in \mathbb{N}, L \geq 1$. If there is a linear subspace $E$ of $\mathbb{R}^n$ such that $A(E) \in E$. Hence there is a decomposition $\mathbb{R}^n = E \oplus C$ and, with respect to it, $A$ has the form*

$$A = \begin{pmatrix} A_E & B \\ 0 & A_C \end{pmatrix}$$

*such that:*

*(i)* $Spec(A_E) \in \{z \in \mathbb{C} \mid Re(z) < 0\}$.

*(ii)* $jSpec(A_E) \bigcap Spec(A_C) = \emptyset \quad$ *for $j = 2 \cdots L$ (non-resonance condition)*

*(iii)* $Spec(-A) + (L+1)Spec(A_E) = \{-\lambda + \mu_1 + \cdots \mu_{L+1} \mid \lambda \in Spec(A)$ *and* $\mu_1 \cdots \mu_{L+1} \in Spec(A_E)\} \subset \{z \in \mathbb{C} \mid Re(z) < 0\}$ *(non-resonance condition)*

*There exist a real-analytic map $K : U_1 \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$ where $U_1$ is an open neighborhood of 0 in $\mathbb{R}^d$, and a polynomial $R : U_1 \longrightarrow U_1$ of the degree at most L, such that*

$$\begin{aligned} &\mathcal{X} \circ K = DK \cdot R \quad \text{in } U_1, \\ &K(0) = 0, \quad DK\mathbb{R}^d = E, \\ &R(0) = 0, \quad DR(0) = A_E. \end{aligned}$$

**Remark 1.6.** Even the eigenvalue $\lambda$ in modul is great than 1, we can use the Theorem 1.1 with $\frac{1}{\lambda}$. Then, there exists a parameterization $K$ for the instable manifold.

As we mentioned before that the strategy to prove the parameterization method for flows will be similar to the case for maps and will be found in the Chapter 4. Firstly, we write the coefficients of $K$ and $R$ that are expanded in power seires. Then we refomulate $K$ as $K = K_{\leq} + K^{<}$ and find the functional equation for the case of flows in an approriate Banach space. Note that we will need to differential in Banach space whose dimension can be finite or infinite. The chapter 3 we will show the differentiability in Banach spaces.

In the chapter 5, the parameterization method has applied to Lorenz system. Therefore, we can compute 2D stable manifold of the Lorenz system at the origin. There are three fixed points which $\lambda_1, \lambda_2$ whose the real part are negative. Applying the parameterization method to E generated by the eigenvectors of $\lambda_1$ and $\lambda_2$ at origin, we get the 2D stable manifold of the Lorenz system. In addition, we will interoduce the integration method so that we are able to drawn the Lorenz attractor. The details will be in the section Lorenz system of the chapter 5.

# Chapter 2

# Banach spaces and maps

In this chapter, we will remind some basic facts on Banach Spaces in order to understand the following chapters (see [AMR12]). For instance we will recall some properties of normed spaces, definition of Banach spaces, etc.

## 2.1 Normed Spaces

**Definition 2.1.** *Let $E$ be a vector space over the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. A map $\|\cdot\| : E \longrightarrow \mathbb{R}$ is a norm if it satisfies the following properties:*

*(i)* $\forall x \in E \quad \|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0$.

*(ii)* $\forall \lambda \in \mathbb{K}$ *and* $\forall x \in E \quad \|\lambda x\| = |\lambda| \cdot \|x\|$

*(iii) Triangle inequality:*

$$\forall x, y \in E, \quad \|x + y\| \leq \|x\| + \|y\|$$

*Note that without the condition $\|x\| = 0$ if and only if $x = 0$, $\|\ \|$ is a seminorm.*

**Definition 2.2.** *A normed space $(E, \|\ \|)$ is a vector space $E$ possessing a norm $\|\ \|$.*

**Example 2.3.** Some examples in finite-dimensional spaces:

(i) $(\mathbb{R}^n,\ \|\ \|_p)$ is a normed space for all 1≤ p $< \infty$, where $\|x\| := (\sum_{k=1}^{n} x^p)^{\frac{1}{p}}$.

(ii) $(\mathbb{R}^n,\ \|\ \|_\infty)$ is a normed space, where $\|x\|_\infty := \max(x_1, \cdots, x_n)$.

The norm $\|\cdot\|$ induces a distance $d(x, y) = \|x - y\|$, so that $(E, d)$ is a metric space. It is also a topological space, being the balls

$$B(x, r) = \{y \in E \quad | d(x, y) < r\}$$

form a basis of neighborhoods of the topology.

**Definition 2.4.** *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space E are called equivalent if they induce the same topology on E.*

**Proposition 2.5.** *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space E are equivalent if only if there exist $\lambda, \beta$ > 0 such that*

$$\lambda \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1 \tag{2.1}$$

In the finite-dimensional case, all the norms are equivalent. On the other hand, an infinite-dimensional space can have many different unequivalent norms.

**Definition 2.6.** *A normed space (E, $\|\cdot\|_E$) over the field $\mathbb{K}$ is a Banach space if E is complete, i.e, for every Cauchy sequence $\{x_n\}$ in E there exists a element x in E such that*

$$\|x_n - x\|_E \underset{n\to\infty}{\longrightarrow} 0$$

**Example 2.7.** The following examples are Banach spaces

(i) Usually, the notion of Banach is used in the infinite dimensional space, in particular use as a vector space of continuous functions.

$$(C([a,b]),\ \|f\| = \sup_{x\in[a,b]} |f(x)|) \text{ is a Banach space}$$

where $C([a,b]) = \{f \quad | \ f$ is a map such that continuos in $[a,b]\}$.

(ii) For each 1 $\leq$ p $<\infty$, ($\ell^p$, $\|\cdot\|_p$) is a Banach space where

$$\ell^p = \{x = \{x_n\}_n \ \mid \sum_{n=0}^{\infty} |x_n|^p < \infty\},$$
$$\|x\|_p := (\sum_{n=0}^{\infty} |x_n|^p)^{\frac{1}{p}}$$

For p = $\infty$, ($\ell_\infty$, $\|\cdot\|_\infty$) also is a Banach space, where

$$\ell_\infty = \{x = \{x_n\}_n \ |x \text{ is a bounded sequence}\}$$
$$\|x\| = \sup_n |x_n| < \infty$$

(iii) Let $H = \{f : \bar{D} \subset \mathbb{R} \to \mathbb{R} \mid f(z) = \sum_{n=0}^{\infty} f_n z^n, \|f\|_H := \sum_{n=0}^{\infty} |f_n| < \infty\}$ where $\bar{D}$ is the closed unit disk centered at origin, then (H, $\|\cdot\|_H$) is a Banach space. Moreover, $f \in H$ that means $f$ is analytic in D and continuous in $\bar{D}$

*Proof.* It suffices to prove any Cauchy sequence $\{f^k\}_k$ in H converges in H. In H we also consider $\|f\|_\infty = \max_{z\in\bar{D}} |f(z)|$, but bearing in mind $\|\cdot\|_\infty$ and $\|\cdot\|_H$ are not equivalent. Since $\{f^k\}_k$ is a Cauchy sequence and $\left\|f^k\right\|_\infty \leq \left\|f^k\right\|_H$ for $f^k \in$ H, $f^k$ is Cauchy in H with $\overline{B(0,\rho)}$ and $\|\cdot\|_\infty$ for all $\rho < 1$. Notably, (H,$\|\cdot\|_\infty$) is a Banach space then there exists $f \in \overline{B(0,\rho)}$ for

all $\rho \leq 1$ such that $f^k \underset{k\to\infty}{\to} f$ Hence, $\{f^k\}_k$ converges uniformly on compact sets $\subset D$ and for all z $\in \overline{D}$ $f^k(z) \underset{k\to\infty}{\longrightarrow} f(z)$ .
Now, we want to illustrate that $f \in H$ and $f^k \underset{k\to\infty}{\longrightarrow} f$ in H where $f^k(z) = \sum_{n=0}^{\infty} f_n^k z^n$. We have $\forall \epsilon > 0\ \exists k_0$ such that $\forall k \geq k_0\ \forall p$

$$\forall m, \quad \sum_{n=0}^{m} \left| f_n^{k+p} - f_n^k \right| \leq \sum_{n=0}^{\infty} \left| f_n^{k+p} - f_n^k \right| = \left\| f^{k+p} - f^k \right\|_H < \epsilon$$

so that

$$\forall m, \quad \lim_{p\to\infty} \sum_{n=0}^{m} \left| f_n^{k+p} - f_n^k \right| = \sum_{n=0}^{m} \left| f_n - f_n^k \right| \leq \epsilon \tag{2.2}$$

Letting $m \to \infty$ we get $\sum_{n=0}^{\infty} \left| f_n - f_n^k \right| \leq \epsilon$ and $f - f^k \in H$.

$$\left\| f - f^k \right\|_H = \sum_{n=0}^{\infty} \left| f_n - f_n^k \right| \leq \epsilon$$

Since $f - f^k \in$ H, $f^k \in$ H and H is vector space then $f = (f - f^k) + f^k \in$ H. In addition

$$\lim_{k\to\infty} \left\| f - f^k \right\|_H = 0$$

$\square$

**Definition 2.8.** *$\mathcal{A}$ is a Banach algebra if $\mathcal{A}$ sastifies the following properties:*

*(i)* *$\mathcal{A}$ is an algebra over the field $\mathbb{K}$.*

*(ii)* *$\mathcal{A}$ is Banach space with norm $\|\cdot\|$.*

*(iii)* $\|xy\| \leq C\,\|x\| \cdot \|y\|$.

**Remark 2.9.** Suppose we have a equivalent norm $\|\cdot\|' = k \cdot \|\cdot\|$ such that

$$\|xy\|' = k \cdot \|xy\| \leq k \cdot C \frac{\|x\|'}{k} \cdot \frac{\|y\|'}{k}$$

If $k = C$, then $\quad \|xy\|' = \|x\|\,\|y\|$

**Example 2.10.** Namely (iii) of Example 2.7 is a Banach algebra

*Proof.* Firstly, H is a vector space over $\mathbb{C}$ with addition and multiplication operation that are closed in H.Immediately, the first statement holds.

Secondly,

$$\begin{aligned}\|f \cdot g\|_H = \sum_{n=0}^{\infty} |(f \cdot g)_n| &= \sum_{n=0}^{\infty} \left| \sum_{n_1+n_2=n} f_{n_1} \cdot g_{n_2} \right| \leq \sum_{n=0}^{\infty} \sum_{n_1+n_2=n} |f_{n_1}| \cdot |g_{n_2}| \\ &= \sum_{n_1=0}^{\infty} |f_{n_1}| \cdot \sum_{n_2=0}^{\infty} |g_{n_2}| = \|f\|_H \cdot \|g\|_H \,.\end{aligned}$$

Finally, H is complete using the (iii) of the example 2.7 $\square$

## 2.2 Continuous linear and multilinear maps

In this subsection we will deal with linear or multilinear maps. We begin with a brief summary of important facts without proofs. Let E, F, G be normed vector spaces over the field $\mathbb{K}$ throughout this subsection.

**Definition 2.11.** *$T : E \to F$ is a linear map if*

$$T(\lambda x + \mu y) = \lambda T x + \mu T y \quad \textit{for all } \lambda, \mu \in \mathbb{K}$$

*$T : E_1 \times \underset{k}{\cdots} \times E_k \to F$ is a k-linear map if T is linear in each component, that is , for $i = 1, \cdots, k$ if $x_i = \sum_{j=1}^{n_i} \lambda_{ij} y_j$*

$$T(x_1, \cdots, x_k) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} \lambda_{1i_1} \cdots \lambda_{ki_k} f(y_{j_1}, \cdots, y_{j_k})$$

**Theorem 2.12.** *Let T: $E \to F$ be a linear maps, recall that the following are equivalent:*

*(i)* *T is continuous at $0 \in E$.*

*(ii)* *T is continuous.*

*(iii)* *T is bounded, that is, there exists a constant $C < \infty$ such that $\|Tx\|_F \leq C \|x\|_E$ for all $x \in X$.*

*In this case, we can define the norm of T by*

$$\|T\| = inf\{C > 0 \mid \textit{satisfying (iii)}\}$$

*is equivalent to*

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{\|x\|_X \leq 1} \|Tx\|_F$$

**Definition 2.13.** *Let $f : E_1 \times \underset{k}{\cdots} \times E_k$ k-linear map and $E_1, \cdot, E_k$ normed spaces. $f$ that are continuous, that is, for each $f \in L(E_1, \cdots, E_n; F)$ there exists C > 0 such that for any $(x_1, \cdots, x_n) \in E_1 \times \cdots \times E_n$*

$$\|f(x_1, \cdots, x_k)\| \leq C \|x_1\|_{E_1} \|x_2\|_{E_2} \cdots \|x_k\|_{E_k} \tag{2.3}$$

*In addition, the operator norm is the smallest C satisfying the condition 2.3.*

To avoid the confusion, we use $\|\cdot\|_E$ or $\|\cdot\|_F$ to identify the norms to which belong.

**Definition 2.14.** *Let $E_1, \ldots, E_k$ be k normed spcaes, F a normed space. We difine $L(E_1, \cdots, E_k; F)$ the spaces of multilinear or k-linear maps from $E_1 \times \cdots \times E_k$ into F that are continuous. If $E_1 = E_2 = \cdots = E_k = E$, the multilinear space will be simply be denoted by $L^k(E; F)$.*

In particular k = 0, we define $L^0(E; F) \cong F$ and k = 1, we denote $L(E; F)$ the space of continuous linear maps from E to F. For each $f \in L(E; F)$ can be normed by its operator norm as we mentioned in Theorem 2.12.

For k > 1 we denote $L(E_1 \times \underset{k}{\cdots} \times E_k; F)$ the space of continuous k-linear maps from $E_1 \times \underset{k}{\cdots} \times E_k$ into F. For each $f \in L(E_1 \times \underset{k}{\cdots} \times E_k; F)$ is a K-linear map that is linear separately in each variable and its operator norm $\|\cdot\|$ is bounded.

$$\begin{aligned}\|f\| &= \inf\{C \mid \|f(x_1, \cdots, x_k)\|_F \leq C\,\|x_1\|_{E_1}\,\|x_k\|_{E_k} \quad \forall (x_1, \cdots, x_k) \in E_1 \times \underset{k}{\cdots} \times E_k\} \\ &= \underset{\|x\|_{E_1} \leq 1, \cdots \|E_k\| \leq 1}{sup} \|f(x_1, \cdots, x_k)\|_F = \underset{x_1 \neq 0, x_k \neq 0}{sup} \frac{\|f(x_1, \cdots, x_k)\|_F}{\|x_1\|_{E_1} \cdots \|x_k\|_{E_k}} < \infty\end{aligned}$$

**Remark 2.15.** Given $f \in L(E_1, \cdots, E_m, \cdots, E_{m+k}; F)$,for $x_1 \in E_1, .., x_m \in E_m$ we define $\tilde{f} \in L(E_{m+1}, \cdots, E_{m+k}; F)$ as

$$\tilde{f}(x_1, \cdots, x_m) : (x_{m+1}, \cdots, x_{m+k}) \to f(x_1, \cdots, x_{m+k})$$

. Note that $\tilde{f} \in L(E_1, \cdots, E_m; L(E_{m+1}, \cdots, E_{m+k}; F))$. Moreover, it is an isomoprhism of vector space which preserve norms, i.e, $\|f\| = \|\tilde{f}\|$
Hence,

$$L(E_1, \cdots, E_m, \cdots, E_n) \cong L(E_1, \cdots, E_m; L(E_{m+1}, \cdots, E_{m+k}; F))$$

In particular, if $E_1 = E_2 = \cdots = E_{m+k} = E$, we obtain

$$L^{m+k}(E; F) \cong L^m(E; L^k(E; F)) \tag{2.4}$$

Immediately, we can observe that $L^k(E; F)$ forms a subspace of the vector space of all linear maps from $E \times \cdots \times E$ into F.

**Proposition 2.16.** *F is a Banach space, then $L(E_1, \cdots, E_n; F)$ is a Banach space with the operator norm defined in Theorem 2.12. (in book [AMR12] gives a proof for case n = 1)*

We define a subspace $L^k_s(E; F)$ of $L^k(E; F)$. Let $S_k$ denote the permutation group on k elements that consists of all bijections $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$

**Definition 2.17.** *The vector space $L^k_s(E, F)$ consists of the symmetric k-linear maps from $E^k$ into F, in others words, for each $f \in L^k(E; F)$ sastisfies $f^\sigma = f$ where*

$$f^\sigma(x_1, \cdots, x_k) := f(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

*that is*

$$f(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = f(x_1, \cdots, x_k) \quad \textit{for any permutation } \sigma \textit{ of } S_k$$

**Remark 2.18.** Note that $L^k_s(E, F)$ is closed in $L^k(E, F)$, thus if F is a Banach space, so is $L^k_s(E, F)$.

Our next considerations will recall a few facts related to homogeneous polynoimal.

**Definition 2.19.** *A homogeneous polynomial map of degree k from E into F (or simply k-homogeneous polynomial) is a map $\varphi : E \to F$ induced by $f \in L^k(E; F)$ and defined by*

$$\varphi(x) = f(x, \cdots, x) \quad \forall x \in E$$

We define that $S^k(E; F)$ is formed by the homogeneous polynomial of degree k from $E$ to $F$. Furthermore,

$$|\varphi(x)|_F \leq |f(x, \cdots, x)|_F \leq \|f\| \cdot |x|^k$$

**Remark 2.20.** Let E = $\mathbb{C}^n$ a finite-dimensional space, then $S^k(E;F)$ is just as a usual homogeneous polynomial of degree k in the n coordinates of $x$ with coeffcients in F. We use $\mathbb{C}^n_k[x] := S^k(\mathbb{C}^n;\mathbb{C})$ to emphasize that E is a complex space.

$$\mathbb{C}^n_k[x] = \{P \mid P(x) = \varphi(x) = \sum_{|m|=k} p_m x^m\} \tag{2.5}$$

where we are using the multi-index notation, that is, $m = (m_1, \cdots, m_k)$ such that $m_1 + \cdots + \mathfrak{m}_k = k$ and $x^m = x^{m_1} \cdots x^{m_k}$.

**Remark 2.21.**

(i) Note that there is an interesting fact about the operator norm of $P \in S^k(E;F)$. Assuming that the norm $|x|$ = max$(x_{1,\dots}, x_k)$ for x = $(x_1, \cdots, x_k) \in$ E

$$|P(x)| \leq \sum_{|m|=k} |p_m| \cdot |x_1|^{m_1} \cdots |x_k|^{m_k} \leq (\sum_{|m|=k} |p_m|)\, |x|^k$$

so that

$$\|P\| \leq \sum_{|m|=k} |p_m|$$

In fact, $\|P\| = \sum_{m=|x|} |p_m|$.

(ii) Using the same strategy we will define the operator norm of $f \in L^k(E;F)$. Considering $\{e_1, \cdots, e_n\}$ a basis of E, the images of f are determined in the following way. If $x_i = \sum_{j=1}^n x_{ij} e_j$ for $1 \leq i \leq k$ and using $f$ multilinearity we obtain

$$f(x_1, \cdots, x_k) = \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n x_{1i_1} \cdots x_{ki_k} f(e_{i_1}, \cdots, e_{i_k}).$$

Then, we associate $f$ to the following norm

$$|f|_k := \sum_{i_1, \cdots, i_n=1}^n \|f(e_{i_1}, \cdots, e_{i_k})\|_F$$

In fact, $|\cdot|_k$ is actually a norm and is equivalent to the operator norm $\|f\|$.

(iii) For any f belongs to $L^k(E;F)$, there will be a $g \in L^k_s(E;F)$ defined by

$$g(x_1, \cdots, x_k) := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} f(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

$g$ is usually to be named $Sym_k(f)$ and

$$|g| = |Sym_k f| = \left| \sum_{\sigma \in \mathcal{S}_k} \frac{1}{k!} f^\sigma \right| \leq \sum_{\sigma \in \mathcal{S}_k} \frac{1}{k!} |f^\sigma| = \sum_{\sigma \in \mathcal{S}_k} \frac{1}{k!} |f| = |f|$$

Furthermore, given $\varphi \in S^k(E, F)$ there is only one $g \in L^k_s(E;F)$ such that $\varphi(x) = g(x, \cdots, x)$ and $\forall x \in E, \quad |\varphi(x)|_F = |g_\varphi(x, \cdots, x)| \leq \|g_\varphi\| \cdot |x|^k$.Hence, we can define in $S^k(E;F)$ $\|\varphi\| := \|g_\varphi\|$

**Definition 2.22.** *The spaces $L_s^k(E;F), k = 1,2,\cdots$ is said to be consistently normed if for k > 0 the space $L_s^k(E;F)$ has a norm $|\cdot|_k$ such that hold the following properties:*

*(i)* $\{L_s^K(E;F), |\cdot|\}$ *is a Banach space.*

*(ii)* $\|f(x_1,\cdots,x_k)\|_F \leq |f|_k \cdot \|x_1\|_E \cdots \|x_k\|_E$, *for all* $x_i \in E$.

*(iii) The isomorphism of* $L^{m+k}(E;F) \cong L^m(E;L^k(E;F))$ *is norm-preserving.*

## 2.3 Differentiability in Banach spaces

In this section we will run through some definitions that is needed in the following chapters such as the definition of the differentiable function, the derivative of $f$ at point $u_0$. Indeed, The converse of Taylor's Theorem and The Implicit Function Theorem will be stated this section.

Let E, F, G be normed vector space over the field $\mathbb{K}$ throughout Chapter 2.

**Definition 2.23.** *Assuming $f, g : U \subset E \to F$ where U is a open set in E. We say g is tangent to f at the point $x_0 \in U$ if*

$$\lim_{x \to x_0} \frac{\|f(x) - g(x)\|}{\|x - x_0\|} = 0$$

**Definition 2.24.** *We say $f : U \subset E \to F$ is differentiable at $u_0$ if only if there exists a continuous linear map $Df(u_0) : E \to F$ such that*

$$\lim_{h \to 0} \frac{\|f(u_0 + h) - f(u_0) - Df(u_0)(h)\|}{\|h\|} = 0 \tag{2.6}$$

*If f is differentiable at each $u_0 \in U$, the map*

$$\begin{aligned} Df :& U \to L(E;F) \\ & u_0 \to Df(u_0) \end{aligned}$$

*is said to be the differential of f and the evaluation $Df(u_0)$ on $v \in E$ will be denoted $Df(u_0)e$.*

*If $Df$ is differential and also a continuous map (satisfying the definition 2.11) we say f is continuously diferentiable and denote $f \in \mathcal{C}^1(U;F)$. We can get the second derivate of f and denote $D^2 f := D(D(f))$ provided that $Df$ is differentiable.*

$$\begin{aligned} D^2 f :& U \to L(E; L(E:F)) \\ & u_0 \to D^2 f(u_0) \end{aligned}$$

*Using 2.4 we can consider $D^2 f$ as a map fromm U into $L^2(E;F)$ and if $D^2 f$ turns out continuous f is said to be of class $\mathcal{C}^2(U;F)$.*

*Proceeding inductively, the n-th derivative of f is a consequence of existing a map*

$$D^n f : U \to L^n(E : F)$$

*and if $D^n f$ is a continuous map we say f is of class $C^n(U;F)$.*

**Remark 2.25.** Schwarz Lemma states that $D^n f \in L^n_s(E;F)$. Namely, if $Df(x)$ exists for $x \in U$, then $Df^2(x)$ is symmetric ,that is,

$$D^2 f(x)(u,v) = D^2 f(x)(v,u) \quad \text{for all } (u,v) \in E \times E$$

Hence, $D^n f : U \to L^n_s(E;F)$ for n $\geq 2$. (The proof will be found in [Car77]).

**Definition 2.26.** *The symbol $O(g(x))$ or $o(g(x))$ are known as Landau symbol and satisfying:*

$$f : U \subset E \longrightarrow F$$
$$g : U \subset E \longrightarrow \mathbb{C}$$

*(i) if $f(x) = o(g(x))$ as $x \to x_0$ where $x_0 \in U$, it will hold*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

*(ii) $f(x) = O(g(s))$, if $\exists U \ni x_0$, $C > 0$ such that $|f(x)| \leq C \cdot |g(x)|$ $\forall x \in U$.*

**Remark 2.27.** Using Landau symbol, it is clear if $f : U \subset E \to F$ is differentiable at $u_0 \in U$ if only if there exists a linear map $Df(u) \in L(E;F)$ such that

$$f(u+h) = f(u) + Df(u) \cdot h + o(\|h\|)$$

and also we are able to reformulate 2.6 as

$$\|f(u+h) - f(u) - Df(u)(h)\| = o(\|h\|)$$

**Example 2.28.**

(i) Given a continuous linear map $f : E \longrightarrow F$ and let $\widehat{f} := f_{|U}$ where $U \subset E$.
$Df : U \longrightarrow L(E;F)$ is $Df(x) = f$ for all $x \in E$ that means $Df(x)$ is a constant map, that is, $Df(x)v = f(v)$. Moreover, $D^2 f = 0$ due to the fact that the derivative of a constant map is zero.

(ii) Given a continuous bilinear map $f : E \times F \longrightarrow F$. $f$ is differentiable and its derivative at $(u,v) \in E \times F$ is

$$\begin{aligned} Df(u,v) :& E \times F \longrightarrow G \\ & (a,b) \longrightarrow Df(u,v)(a,b) := f(u,b) + f(a,v) \end{aligned}$$

**Definition 2.29.** *Considering $f : U \subset E_1 \oplus E_1 \longrightarrow F$ where $E_1$ and $E_2$ are normed vector space and assuming $u_0 = (u_{01}, u_{02}) \in U$. We denote $D_1 f(u_0) \in L(E_2, F)$ and $D_2 f(u_0) \in L(E_1, F)$ and say they are partial derivatives if for $v_1 \in E_1$ and $v_2 \in E_2$ the derivatives of the maps $v_1 \longmapsto f(v_1, u_{02})$ and $v_2 \longmapsto f(u_{01}, v_2)$ exist, respectively.*

We will give a useful property of partial derivatives in preceding proposition.

**Proposition 2.30.** *Let $U \subset E_1 \oplus E_2$ be a open and $f : U \longrightarrow F$*

*(i) If $f$ is differentiable, then the partial derivatives at $(u_1, u_2) \in E_1 \oplus E_2$ exist and are define by*

$$D_1 f((u_1, u_2)) \cdot e_1 = Df(u) \cdot (e_1, 0)$$
$$D_2 f((u_1, u_2)) \cdot e_2 = Df(u) \cdot (0, e_2)$$

*(ii) If $f$ is differentiable, then*

$$Df((u_1, u_2)) \cdot (e_1, e_2) = D_1 f((u_1, u_2)) \cdot e_1 + D_2 f((u_1, u_2)) \cdot e_2$$

In fact, (ii) of Example 2.28 is a direct consequence of Proposition 2.30. Considering that $Df(u, v) = f$

**Proposition 2.31.** *Considering $f : U \longrightarrow \mathbb{R}^n$ is differentiable wehre $U \subset \mathbb{R}^m$ is an open set. Then the partial derivatives $\frac{\partial f^j}{\partial x^i}$ exist, and the Jacobian matriz of $f$ is given by the matrix of the linear map $Df(x)$ with respect to the standard bases in $\mathbb{R}^n$ in the following form*

$$\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & \frac{\partial f^m}{\partial x^n} \end{bmatrix}$$

*where each partial derivative is evaluated at $x = (x^1, \cdots, x^n)$*

## 2.4 Properties of the Derivative

In this section we will illustrate some fundamental properties of derivatives that will be required later. These properties are analogous of rules familiar from elementary calculus.

(i) **Linearity of the Derivate**

Let $f, g : U \subset E \to F$ be r times differentiable maps and $\lambda$ real or complex constant ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ ).Then $\lambda f$ and $f + g : U \subset E \to F$ are r times differentiable such that

$$D^r(f + g) = D^r f + D^r g \text{ and } D^r(\lambda f) = \lambda D^r f$$

(ii) **Derivative of a Certesian Product**

Assuming $F_i$ normed vector spaces and $f_i : U \subset E \to F_i$, $1 \leq i \leq n$ r times differentiable maps. Then

$$f = (f_1 \times \cdots \times f_n) : U \subset E \longrightarrow F_1 \times \cdots \times F_n$$
$$u_0 \longrightarrow f(u_0) = (f_1(u_0)), \cdots, f_n(u_0))$$

is r times differentiable and $D^r f = D^r f_1 \times \cdots \times D^r f_n$.

(iii) **Composite Mapping Thereom**

Suppose $U \subset E, V \subset F$ , $f : U \longrightarrow V$ and $g : V \longrightarrow G$ are differentiable maps. Then so is the composite $g \circ f : U \longrightarrow G$ and it is

$$D(g \circ f)(u) = D(g(f(u))) \circ D(f(u))$$

## 2.5 Taylor's formula and the converse Taylor's Theorem

In this section, the main idea is to turn out the Taylor's formula and the converse Taylor's Theorem so as to be used in later chapters. Firstly, we state Taylor's formula.

**Theorem 2.32.** *Let E, F be Banach spaces and U an open set in E. Assuming that $f : U \longrightarrow F$ is n times differentiable at $a \in U$. Then for h such that $a + h \in U$*

$$f(a+h) = f(a) + \frac{Df(a)}{1!} \cdot h + \frac{D^2 f(a)}{2!} \cdot h^2 + \cdot + \frac{D^n f(a)}{n!} \cdot h^n + o(\|h\|^n)$$

*where $D^n f(a) \cdot h^n = D^n f(a)(h, \overset{n}{\cdots}, h)$.*

Note that $f(a+h)$ is approximated by a polynomial of degree at most n in F. Now, we will state thw so-called converse to Talyor's Theorem.

**Theorem 2.33.** *Given a map $f : U \subset E \longrightarrow F$ such that there are continuous maps*

$$\varphi_k : U \subset E \longrightarrow L^k(E; F), \quad k = 1, \cdots, r$$

*such that for all $a \in U$ and $a + h \in U$, sastifying*

$$f(a+h) = f(a) + \frac{\varphi_1(a)}{1!} \cdot h + \frac{\varphi_2}{2!} \cdot h^2 + \cdots + \frac{\varphi_n}{n!} \cdot h^n + o(\|h^n\|).$$

*then $D^k f(a) = \varphi_k(a)$, for all $k = 1, \cdots, r$.*

## 2.6 The Implicit Function Theorem

The last Theorem we will state in this chapter is the so-called Implicit Function Theorem in Banach spaces. It is an important tool to prove the Parameterization Method of invariant manifolds. Let us see how powerful is this Theorem.

**Theorem 2.34.** *Let E, F, G Banach spaces and $U \subset E, V \subset F$ be open in E and F respectively. Assume $f : U \times V \longrightarrow G$ be a $C^r$ map and for $(x_0, y_0) \in U \times V$, $D_2 f(x_0, y_0) : F \longrightarrow G$ is an isomorphism. Then there exist neighbourhoods $U_0$ of $x_0$, $W_0$ of $f(x_0, y_0)$ and a unique $C^r$ map $g : U_0 \times W_0 \longrightarrow V$ such that for all $(x, w) \in U_0 \times W_0$*

$$f(x, g(x, w)) = w$$

Note that $D_2 f(x_0, y_0)$ is isompophism, the invers is also continous.

**Remark 2.35.** Pay attetion when we fix $w$ = 0, which is the case that Theorem 2.34 is often used. Using the Theorem 2.34 exist $U_0, V_0$ to which $x_0$, $y_0$ belong and $g : U_0 \to V_0$ such that

$$\{(x, y) \in U_0 \times V_0 \mid f(x, y) = 0\} = \{x, y = g(x)\} \tag{2.7}$$

that means $y$ can be locally solved in $U_0$. Furthermore, g as regular as f is.

In fact, 2.7 give the uniqueness of solutions. Let $(x, y)$, $(x, \hat{y}) \in U_0 \times V_0$ two solution $f(x, y) = 0, f(x, \hat{y})$, then 2.7 tell us that $y = g(x) = \hat{y}$.

# Chapter 3

# Banach spaces of analytic functions

Let E, F be banach spaces that both are real or both are complex throughout this chapter. To avoid the confussion we will use the subscripts in the norms such as $|\cdot|_E$ to E. We will introduce the spaces $A_\delta(E;F)$ and some properties in order to prove the so-called Omega Lemma, which is an important tool for the next chapter.

## 3.1 Analytic Functions and the spaces $A_\delta(E;F)$

We consider the family of norms $|\cdot|_k$ in $L^k(E;F)$ for instance using $|\cdot|_k$ the norms induced by the norms in E, F that turned up in Remark 2.21, but for simplicity from now on we will omit the subscripts in the norms of $L^k(E;F)$. Just keep in mind that there is a family of norms for each $L^k(E;F)$ which is consistent. This section we will deal with the analytic functions in Banach spaces and demonstrate that this fuctions satisfy some similar properties as usual.

**Definition 3.1.** *Let $U \in E, f : U \longrightarrow F$ f is said to be an analytic function at the point $x_0 \in U \subset E$ if there exists $\delta > 0, a_k \in L_s^k(E;F)$ such that*

$$\forall x \in B(x_0)_\delta \quad f = \sum_{k=0}^{\infty} a_k(x-x_0)^{\otimes k} \quad \textit{and} \quad \sum_{k=0}^{\infty} |a_k|\,|x-x_0|^k < \infty$$

*where $a_0 \in F$ and for k > 0 $a_k$ is a k-linear, symmetric map from $E^k$ to F, i.e, $a_k \in L_s^k(E;F)$. We say $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series in $(x-x_0)$ with values in F. Furthermore, if a power series with only a finite number of terms is a polynomial.*

Since we stated the definition of analytic functions, we now introduce the space where we will deal with in the following chapter.

For a $\delta > 0$, we consider $A_\delta(E;F)$ the space of all power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ such that $\sum_{k=0}^{\infty} |a_k|\,\delta^k < \infty$. The norm for each $f$ is defined by $\|f\|_\delta = \sum_{k=0}^{\infty} |a_k|\,\delta^k$ and it turns out that $A_\delta(E;F)$ is a vector space with addition and product by scalars as usual. Morever, $A_\delta(E;F)$ endowed with $\|\cdot\|_\delta$ is a Banach space. The completeness of this space is proved similar to the proof showed up in (iii) of Example 2.7, but the leading idea is same.

**Remark 3.2.** It $f \in A_\delta(E;F)$, then $f$ is absolutely and for each $|x| \leq \delta$ uniformly convergent in $\bar{B}_{\delta(0)}$ by Weierstrass M-test. Thus, f is continuous in $\overline{B(0,\delta)}$ so obviously using the definition of continuity $|f(x)| \leq \|f\|_\delta$ whichever $|x|_E$ and for each term $a_k\delta^k \leq \|f\|_\delta$. Hence, $\|f\|_\infty \leq \|f\|_\delta$

Observe that if $0 < \rho < \delta$

$$A_\delta(E;F) \subset A_\rho(E;F) \quad \text{and} \quad \|f\|_\rho \le \|f\|_\delta$$

From now on, for any $f \in A_\delta(E;F)$ we turn around the map

$$f : \overline{B(0,\delta)} \subset E \longrightarrow F$$

In particular, the following proposition will give a explicit upper bound for $\|\cdot\|_\rho$ in case E = $\mathbb{C}^n$.

**Proposition 3.3.** *If $f : \{x \in \mathbb{C}^n \mid |x_i| < \delta\} \longrightarrow F$ is analytic and bouded in norm by M, then for each $\rho \in (0,\delta)$, $f \in A_\rho(\mathbb{C}^n;F)$ and $\|f\|_\rho \le M(1-\frac{\rho}{\delta})^{-n}$.*

Now, we are interested in the case $E = \mathbb{R}^n$or $\mathbb{C}^n$. Since E is finite-dimensional, we can pick a orthonormal basis $\{e_1, \cdots, e_n\}$ in E. Then, for $x \in E$ we can write $x = \sum_{j=1}^n x_j e_j$ and for each $a_k \in L_s^k(E;F)$ we will state an explicit formula

$$\begin{aligned} a_k x^k &= \sum_{i_1,\cdots,i_k=1}^{n} a_k(e_{i_1}, \cdots, e_{i_k}) x_{i_1} \cdots x_{i_k} \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_n!} a_k(\underbrace{e_1, \cdots, e_1}_{\alpha_1}, \cdots, \underbrace{e_n, \cdots, e_n}_{\alpha_n}) x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n} \end{aligned}$$

where the first equality is obtained by $a_k$ is k-linear. The second is got by its symmetric and $\alpha_i$ stands for how many times $e_i$ occurs. Observe that each $a_k x^k$ is a homogeneous polinomial of the form $\sum_{|m|=k} p_m x^m$ as we mentioned in 2.5 and its operator norm showed up in (ii) of Remark 2.21.

Let us state some propositions and lemmas will be helpful in later.

**Lemma 3.4.** *Consider $b_k \ge 0$, k positive integer and $\delta > 0$ such that $\sum_{k=0}^\infty b_k x^k = M < \infty$. Then for any positive integer j and $\rho < \delta$*

$$\sum_{k=j}^{\infty} \binom{k}{j} b_k \rho^{k-j} \le \frac{M}{(\delta-\rho)^j}$$

**Theorem 3.5.** *If $f \in A_\delta(E;F)$ then, for all $i \ge 1$, there exists $D^i f$. Moreoever, for $0 < \rho < \delta$ $D^i f \in A_\delta(E; L_s^i(E;F))$ and*

$$\left\| D^i f \right\|_\rho \le \frac{i! \, \|f\|_\delta}{(\delta-\rho)^i}$$

*In particular f is $C^\infty$ in the disc $B(0,\delta) = \{x \in E \mid |x|_E < \delta\}$*

*Proof.* Let $f(x) = \sum_{k=0}^\infty f_k x^k$ is analytic in $B_\delta(0)$ and is bounded by M. For $0 < \rho < \delta$ we have

$$\begin{aligned} \infty > \|f\|_\delta = \sum_{k=0}^{\infty} |a_k| \, x^k = \sum_{k=0}^{\infty} |a_k| \, (\rho + (\delta-\rho))^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} |a_k| \, \rho^{k-i} (\delta-\rho)^i \\ = \sum_{i=0}^{\infty} \left( \sum_{k=i}^{\infty} \binom{k}{i} |a_k| \, \rho^{k-i} \right) (\delta-\rho)^i \end{aligned} \tag{3.1}$$

where the third equality is got by binomial expansion and the last equality interchange the summations.Let $|x| < \rho$ and $|y| < \frac{1}{2}(\delta - \rho)$. Then

$$f(x+y) = \sum_{k=0}^{\infty} a_k(x+y)^k = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_k x^{k-i} y^i\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_k(x^{k-i},\cdot)\right) y^i \tag{3.2}$$

where the last equality is hold thanks $L^k(E;F) \cong L^{k-i}(E;L^i(E;F))$. Moreover, the last two series in 3.2 in norm are smaller than the two series in 3.1, which means that the last two series converge absolutely.

Using the Definition 2.6 and 3.2, the $i$th derivatve of f at x ought to be

$$D^i f(x) = i! \sum_{k=i}^{\infty} \binom{k}{i} a_k(x^{k-i},\cdot) \tag{3.3}$$

Let us prove 3.3 is true and $\varphi_i(x)$ denote the right-hand of 3.3.

$$\|\varphi_i(x)\| = i! \sum_{k=i}^{\infty} \binom{k}{i} |a_k| \rho^{k-i} \leq i! \frac{\|f\|_\delta}{(\delta-\rho)^i} \tag{3.4}$$

The last inequality is obtained by Lemma 3.4.

Hence, $\varphi_i(x) \in A_\rho(E; L^i(E;F))$. Then, we reformulate the equation 3.2 in notation of $\varphi_i$

$$f(x+y) = \sum_{i=0}^{\infty} \frac{\varphi_i(x)}{i!}(y^i) = \sum_{i=0}^{n} \frac{\varphi_i(x)}{i!}(y^i) + \sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^i)$$

where

$$\left|\sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^i)\right| = \left|\sum_{i=n+1}^{\infty} \frac{\varphi_i(x)}{i!}(y^{i-n-1},\cdot)(y^{n+1})\right| \leq \left(\left|\sum_{i=n+1}^{\infty} \frac{\|f\|_\delta}{(\delta-\rho)^i}\left(\frac{\delta-\rho}{2}\right)^{i-n-1}\right|\right) \cdot \|y\|^{n+1}$$
$$= \frac{\|f\|_\delta}{(\delta-\rho)^{n+1}} \sum_{j=0}^{\infty} \frac{1}{2^j} \cdot \|y\|^{n+1} = \frac{2\|f\|_\delta}{(\delta-\rho)^{n+1}} \cdot \|y\|^{n+1} = o(\|y\|^n)$$

where the next-to-last equality we use $j = i - n - 1$ and the last equality is hold by geometric series. Finally, applying the converse of Taylor's Theorem we can conculde that $D^i f = \varphi_i$.

It belongs to $A_\rho(E; L^i(E;F))$ and the upper bounded is given by 3.4. □

## 3.2 The Omega-Lemma

The leading idea of this section is to prove the Omega-lemma in the spaces $A_\delta(E;F)$ which demonstrates the operator of the composition of analytic functions is also $C^\infty$ oprator. Let E, F, G be Banach spaces and f, g be functions of $A_\delta(E;F)$ and $A_\rho(G;E)$, respectively.

For each $x \in G$ such that $f(g(x))$ makes sense, it is needed that $|g(x)| \leq \|g\|_\rho < \delta$ and $\|f \circ g\|_\rho \leq \|f\|_\delta$ since we have $|f(g(x))| \leq \|f\|_\delta$. In addition, the power series of $f \circ g$ needs to be a sum of continuous symmetric multilinear maps.

**Remark 3.6.** Given $a_k \in L_s^k(E;F)$ and $b_{j_1}, \cdots, b_{j_k} \in L_s^{j_1}(G;E), \cdots, L_s^{j_k}(G;E)$, in general the composition $a_k(b_{j_1}(\cdot), \cdots, b_{j_k}(\cdot)) \in L^\ell_(G;E)$ where $\ell = j_1 + \cdots + j_k$, but fails to be symmetric. Consider $h(x_1, \cdots, x_\ell)$ using (iii) of Remark 2.21, we get $Sym_\ell(h) = \frac{1}{\ell!}\sum_{\sigma \in \mathcal{S}_l} h(x_{\sigma(1)}, \cdots, x_{\sigma(\ell)})$.

**Lemma 3.7.** *If $g \in A_\rho(E;F)$ with $\|g\|_\rho \leq \delta$ and $f \in A_\delta(E;F)$, then $f \circ g \in A_\rho(E;F)$ and $\|f \circ g\|_\rho \leq \|f\|_\delta$.*

*Proof.* Firstly, $f \circ g$ must be an analytic function where $f, g$ analytic functions, i.e, $f = \sum_{k=0}^{\infty} a_k y^k$ and $g = \sum_{j=0}^{\infty} b_j x^j$ such that their operator norm are bounded. Thus, we need to expand the composition as a power series of continuous symmetric multilinear maps.

For $b_j \in L_s^j(G;E)$ and $a_k \in L_s^k(E;F)$, then $a_k(b_{j_1}(\cdot), \cdots, b_{j_k}(\cdot)) \in L_s^\ell(G;F)$ where $\ell = j_1 + \cdots + j_k$. Therefore, for $(x_1, \cdots, x_\ell)$ the map

$$(x_1, \cdots, x_\ell) \mapsto a_k(b_{j_1}(x_1, \cdots, x_{j_1}), \cdots, b_{j_k}(x_{\ell - k_k + 1}, \cdots, x_\ell))$$

is continuous and $\ell$-linear for due to $a_k$ and $b_j$ are. For $x \in G$

$$\begin{aligned}(f \circ g)(x) &= \sum_{k=0}^{\infty} a_k(\sum_{j_1=0}^{\infty} b_{j_1} x^{j_1}, \cdots, \sum_{j_k}^{\infty} b_{j_k} x^{j_k}) = \sum_{k=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} a_k(b_{j_1} x^{j_1}, \cdots, b_{j_k} x^{j_k}) \\ &= \sum_{l=0} P\infty \sum_{k=0}^{\infty} \sum_{j_1+\cdots+j_k=\ell} a_k(b_{j_1} x^{j_1}, \cdots, b_{j_k} x^{j_k})\end{aligned} \tag{3.5}$$

the next-to-last equality due to the continuit and multilinearity. As we said in Remark 3.6 this map is not symmetric, so let us take $Sym_\ell(h)$ for $h(x) = a_k(b_{j_1} x^{j_1}, \cdots, b_{j_k} x^{j_k})$ where $|Sym_l h| = |h|$ is prooved in (iii) of Remark 2.21. Therefore, we can write 3.5

$$f \circ g = \sum_{\ell=0}^{\infty} \gamma_\ell \quad \text{where} \quad \gamma_\ell = \sum_{k=0}^{\infty} \sum_{j_1+\cdots+j_k=\ell} Sym_\ell a_k(b_{j_1}(\cdot), \cdots, b_{j_k}(\cdot))$$

is symmetric $\ell$-linear and continuous. Note that when $\ell > 0$ the sum $\sum_{j_1+\cdots+j_k=\ell}$ is void for $k = 0$.

$$\begin{aligned}\left|Sym_\ell a_k(b_{j_1} x^{j_1}, \cdots, b_{j_k} x^{j_k})\right| &= \left|a_k(b_{j_1} x^{j_1}, \cdots, b_{j_k} x^{j_k})\right| \leq |a_k| \cdot \left|b_{j_1} x^{j_1}\right| \cdots \left|b_{j_k} x^{j_k}\right| \\ &\leq |a_k| \cdot \left|b_{j_1}\right| \cdot \left|x^{j_1}\right| \cdots \left|b_{j_k}\right| \cdot \left|x^{j_k}\right| = |a_k| \cdot \left|b_{j_1}\right| \cdot \cdots \cdot \left|b_{j_k}\right| \cdot |x|^\ell\end{aligned}$$

where the family of norms on $L_s^k(E;F)$ and $L_s^j(G;E)$ are consitent, then

$$\begin{aligned}\|f \circ g\|_\rho = \sum_{\ell=0}^{\infty} |\gamma_\ell| \rho^\ell &\leq \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_1+\cdots+j_k=\ell} |a_k| \cdot \left|b_{j_1}\right| \cdot \cdots \cdot \left|b_{j_k}\right| \cdot \rho^\ell \\ &= \sum_{\ell=0}^{\infty} \left( \sum_{k=0}^{\infty} |a_k| \sum_{j_1+\cdots+j_k=\ell} \left|b_{j_1}\right| \cdot \cdots \cdot \left|b_{j_k}\right| \right) \cdot \rho^\ell \\ &= \sum_{k=0}^{\infty} |a_k| \left( \sum_{\ell=0}^{\infty} \sum_{j_1+\cdots+j_k=\ell} \left|b_{j_1}\right| \cdot \cdots \cdot \left|b_{j_k}\right| \cdot \rho^\ell \right) \\ &= \sum_{k=0}^{\infty} |a_k| \left( \sum_{j=0}^{\infty} b_j \rho^j \right)^k = \sum_{k=0}^{\infty} |a_k| \cdot \|g\|_\rho^k \leq \sum_{k=0}^{\infty} |a_k| \cdot \delta^k = \|f\|_\delta\end{aligned}$$

□

**Remark 3.8.** If $g(0) = 0$,that is, $b_0 = 0$ then for each $\ell$ the summation is finite

$$\sum_{k=0}^{\infty} \sum_{\substack{j_1+\cdots+j_k=\ell \\ j_1,\cdots,j_k \leq 1}} b_j x^{j_1} \cdots x^{j_k} = \sum_{k=0}^{\ell} \sum_{\substack{j_1+\cdots+j_k=\ell \\ j_1,\cdots,j_k \leq 1}} b_j x^{j_1} \cdots x^{j_k}$$

Before we prove the major goal in this section, we will state the next proposition so as to demonstrate the composition operator in this space of analytic functions is continuous.

**Proposition 3.9.** *Consider $U = \{g \in A_\rho(G;E) \mid \|g\|_\rho =: \alpha < \delta\}$. Then the map*

$$\begin{aligned} \Omega : A_\delta(E;F) \times U &\longrightarrow A_\rho(G;F) \\ (f,g) &\longrightarrow \Omega(f,g) := f \circ g \end{aligned}$$

*is continuous.*

*Proof.* If $(\widetilde{f}, \widetilde{g})$ is close to $(f,g)$ and we prove $\Omega$ is continuous in each argument, then $\Omega$ is continous. Firstly, fixing g we want to see the continuity respect to the first argument. Since we have proved the Lemma 3.7, we have

$$\left\|\Omega(\widetilde{f},g) - \Omega(f,g)\right\|_\rho = \left\|\widetilde{f} \circ g - f \circ g\right\|_\rho = \left\|(\widetilde{f} - f) \circ g\right\|_\rho \le \left\|\widetilde{f} - f\right\|_\delta$$

$\Omega$ is uniformly continuous in the first component independently of the second component.

Now, fixing f let us see what happen with the second component. Denoting $\|g\|_\rho =: \alpha < \delta$ and $\beta := \frac{1}{3}(\delta - \alpha) > 0$. Since $\alpha + \beta = \delta - 2\beta < \delta$, applying Theorem 3.5 we get for $f \in A_\delta(E;F)$, $D^k f \in A_{\delta-2\beta}(E;F)$ and

$$\left\|D^k f\right\|_{\delta-2\beta} \le k! \cdot \|f\|_\delta \cdot (2\beta)^{-k}$$

Choosing a $\widetilde{g} = g + h \in U$ such that $\|h\|_\rho < \beta$,which satisfies

$$\|g+h\|_\rho \le \|g\|_\rho + \|h\|_\rho < \alpha + \beta < \delta.$$

From the Taylor's Theorem we have that for $x \in G$

$$\Omega(f, g+h)(x) - \Omega(f,g)(x) = f(g(x) + h(x)) - f(g(x)) = \sum_{k=1}^{\infty} \frac{D^k f(g(x))}{k!}(h(x)^k) \tag{3.6}$$

Consider $f = \sum_{j=0}^{\infty} a_j, g = \sum_{\ell}^{\infty} b_\ell$ and $h = \sum_{\ell=0}^{\infty} h_\ell$ where $a_j \in L^j(E;F)$ and $g_\ell, h_\ell \in L_s^\ell(E;F)$.Using the explicit formula in 3.3 for $k$th derivate of $f \circ g$ evaluate in $h(x)^k := (h(x), \overset{k}{\cdots}, h(x))$ , we get

$$\begin{aligned} \frac{D^k f(g(x))}{k!}(h(x)^k) &= \sum_{j=k}^{\infty} \binom{j}{k} a_j(g(x)^{j-k}, h(x)^k) \\ &= \sum_{j=k}^{\infty} \binom{j}{k} \sum_{\ell_1, \cdots, \ell_j \ge 0} a_j(b_{\ell_1}(\cdot), \cdots, b_{\ell_{j-k}}(\cdot), h_{\ell_{j-k+1}}(\cdot), \cdots, h_{\ell_j}) x^{\ell_1 + \cdots + \ell_j} \end{aligned} \tag{3.7}$$

where the last equality is obtained by expanding $a_j(g(x)^{j-k}, h(x)^k)$ as what we did in Lemma 3.7 and denote $\kappa = \ell_1 + \cdots + \ell_{j-k} + \ell_{j-k+1} + \cdots + \ell_j$.

$$\begin{aligned} \left\| \frac{D^k f(g(\cdot))}{k!}(h(\cdot)^k) \right\| &\le \sum_{j=k}^{\infty} \sum_{\ell_1, \cdots, \ell_j \ge 0} |a_j| \cdot |b_{\ell_1}| \cdots \left|b_{\ell_{j-k}}\right| \cdot \left|h_{\ell_{j-k+1}}\right| \cdots \left|h_{\ell_j}\right| \rho^\kappa \\ &= \sum_{j=k}^{\infty} \binom{j}{k} |a_j| \cdot \left( \sum_{\ell \ge 0} |b_\ell| \rho^\ell \right)^{j-k} \cdot \left( \sum_{\ell \ge 0} |h_\ell| \rho^\ell \right)^k \\ &= \sum_{j=k}^{\infty} \binom{j}{k} |a_j| \cdot \|g\|_\rho^{j-k} \cdot \|h\|_\rho^k \\ &= \sum_{j=k}^{\infty} \binom{j}{k} |a_j| \cdot \alpha^{j-k} \cdot \|h\|_\rho^k = \left\| \frac{D^k f}{k!} \right\|_\alpha \cdot \|h\|_\rho^k \end{aligned} \tag{3.8}$$

where the last equality is hold by 3.4 in Theorem 3.5

then,

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}\frac{(D^k f)\circ g(\cdot)}{k!}(h(\cdot)^k)\right\|_{\rho} &\leq \sum_{k=1}^{\infty}\left\|\frac{D^k f}{k!}\right\|_{\alpha}\cdot\|h\|_{\rho}^{k} \leq \sum_{k=1}^{\infty}\left\|D^k f\right\|_{\alpha+\beta}\frac{1}{k!}\cdot\|h\|_{\rho}^{k}\\ &\leq \sum_{k=1}^{\infty}\frac{\|f\|_{\delta}}{(2\beta)^k}\cdot\|h\|_{\rho}^{k} \leq \sum_{k=1}^{\infty}\frac{\|f\|_{\delta}}{(2\beta)^k}\cdot\beta^{k-1}\cdot\|h\|_{\rho} \\ &= \frac{\|f\|_{\delta}}{\beta}\cdot\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\|h\|_{\rho} = \frac{\|f\|_{\delta}}{\beta}\cdot\|h\|_{\rho}\end{aligned} \tag{3.9}$$

where the first inequality hold by 3.8, the second because of $\alpha < \alpha+\beta < \delta$ and the third is hold due to $\alpha+\beta < \delta - 2\beta < \delta$ and apply Theorem 3.5. Eventually, 3.6 and 3.9 implies the continuity of $\Omega$ respect to the second component. □

**Theorem 3.10.** *The map $\Omega$ of the previous proposition is $\mathcal{C}^{\infty}$*

*Proof.* We will keep the previous notations throughout this proof. Firstly, we show the existence of the partial derivative to first argument. Observe that $\Omega(f,g) = f\circ g$ is continuous linear in f, so applying what we saw in (i) of Exmaple 2.28

$$D_1\Omega(f,g)[h] = \lim_{s\to 0}\frac{\Omega(f+sh,g)-\Omega(f,g)-\Omega(f,g)}{s} = \lim_{s\to 0} = \frac{f\circ g + s\cdot h\circ g - f\circ g}{s} = h\circ g$$

Therefore, $D_1\Omega(f,g)[h] = h\circ g$ is continuous by Theorem 3.9. we get

$$D_1\Omega(f,g) = \Omega(\cdot,g) \quad \text{and} \quad D_1^2\Omega = D_1^3\Omega\cdots = 0$$

where $D_1\Omega$ is continuous by Proposition 3.9.

Secondly, we will demonstrate of existince of the partial derivative respect to the second argument. Assume $\|h\|_{\rho}$ is small enough.

$$\Omega(f,g+h)(x) = f(g(x)) + \sum_{k=1}^{n}\frac{D^k f(g(x))}{k!}(h(x)^k) + \sum_{k=n+1}^{\infty}\frac{D^k f(g(x))}{k!}(h(x)^k)$$

where the last equality is hold due to $f$ is $\mathcal{C}^{\infty}$ and use the converse Taylor's Theorem. Using the same strategy as in 3.8 and note that

$$\left\|\sum_{k=n+1}^{\infty}\frac{(D^k f)\circ g(x)}{(k)!}(h(x)^k)\right\| \leq \frac{\|f\|_{\delta}}{2^n\beta^{n+1}}\|h\|_{\rho}^{n+1} = o(\|h\|_{\rho}^{n})$$

Since $D^k f \in A_{\delta}(E; L_s^k(E;F))$ and $(D^k(f))\circ g$ is continuous by Lemma 3.7. Then, applying the Converse of Taylor's Theorem we know that $D_2^k(\Omega)(f,g)$ exists and does equal to $D^k f\circ g$ for $0\leq k\leq n$.

Thirdly, let us see the mixed partial derivative. Since $D_2^k\Omega(f,g) = (D^k f)\circ g$ is continuous linear in f, using the same srategy as before

$$D_1 D_2^k\Omega(f,g) = D_2^k(\cdot,g) \quad \text{and} \quad D_1^j D_2^k(f,g) = 0 \quad \text{for } j\geq 2$$

Finally, $\Omega$ has continuous partial derivatives of all orders. Hence, $\Omega$ is $\mathcal{C}^{\infty}$ □

# Chapter 4

# The proofs of the parameterization method

In this chapter we present the proofs of the theorems stated in chapter 1, about the existence of real-analytic invariant manifolds of real-analytic dymanical systems.

## 4.1 The parameterization method for maps

### 4.1.1 Analytic one-dimensional stable manifolds

In this section we prove the theorem that we have mentioned before as a motivation .

*Proof.* of Theorem 1.1
We want to find an analytic map $K : D \subset \mathbb{R} \to \mathbb{R}^n$ suh that

$$F \circ K(s) - K(\lambda s) = 0 \quad \forall s \in D \tag{4.1}$$

Using the technique turn up in Section 1.1 and provided the hypothesis iii, we get

$$K_m = -(A - \lambda^m I)^{-1} E_m \quad \forall m \geq 2 \tag{4.2}$$

where $E_m s^m = [F(K_{<m})(s)]_m$.

We reformulate $K(s) = K_1 s + K^{>}(s)$ where $K_1$ is a multiple of $v$ such that $\|K_1\| = \delta$ and $K^{>}(s) = \sum_{m=2}^{\infty} K_m s^m$ and $F(x) = A(x) + N(x)$ where A is linear part and N is the non-linear part of F. If we replace K(s) in (4.1), we will obtain an equation below for $K^{>}(s)$.

$$AK^{>}(s) + N(K_1 s + K^{>}(s)) - K^{>}(\lambda s) = 0 \tag{4.3}$$

We consider

$$K^{>} \in H^d_{\geq 2} = \{K^{>} : \overline{D} \subset \mathbb{C} \longrightarrow \mathbb{C}^d \mid K^{>}(s) = \sum_{m=2}^{\infty} K_m s^m, \|K^{>}\| = \sum_{m=2}^{\infty} |K_m| < \infty\},$$

where $(H^d_{\geq 2}, \| \, \|)$ is a Banach space of analytic functions in the closure of unit disk, vanishing at the origin along with their first derivative.

Morever, $(H^1_{\geq 0}, \| \, \|)$ is a Banach algebra (proved in example 2.10) and so is the ideal $H^1_{\geq 2} = \{f \mid f_0 = f_1 = 0\}$.
We can rewrite (4.3) as an operator equation in order to prove K as a power series converges.

$$\mathcal{T}(K_1, K^>) = 0 \tag{4.4}$$

where $\mathcal{T} : V \subset \mathbb{C}^n \times H^n_{\geq 2} \longrightarrow H^n_{\geq 2}$ defined by

$$\mathcal{T}(K_1, K^>)(s) := (\mathcal{S}K^>)(s) + N(K_1 s + K^>(s)) \tag{4.5}$$

where $(\mathcal{S}\Delta)(s) = A\Delta(s) - \Delta(\lambda s)$ and $V$ is small neighborhood of (0,0) will be determined later.
Let us comment some main properties of $\mathcal{T}$ in the following propositions: $\square$

**Proposition 4.1.** *If V is contained in a ball of $\mathbb{C}^n \times H^n_{\geq 2}$ centered at (0, 0) and suffiiently small radii, then:*

*(i) The operator $\mathcal{T} : V \subset \mathbb{C}^n \times H^n_{\geq 2}$ is well defined and $C^\infty$.*

*(ii) $D_2\mathcal{T}(0,0) = \mathcal{S}$.*

*Proof.* We can reformulate (4.5) using the $\Omega$ defined in Proposition 3.9

$$\mathcal{T}(K_1, K^>) = \mathcal{S}\Delta + \Omega(N, K_1 + K^>) \tag{4.6}$$

Hence, the first statement is a direct consequence of the Omega-Lemma. In addition, the second partial derivative of $\mathcal{T}$ is

$$D_2\mathcal{T}(K_1 + K^>)\Delta = \mathcal{S}\Delta + D_2\Omega(N, K_1 + K^>)\Delta = \mathcal{S}\Delta + DN(K_1 + K^>)\Delta$$

where $D_2\Omega(N, K_1 + K^>) = DN(K_1 + K^>)$ is proved in the proof of Theorem 3.10. Therefore,

$$D_2\mathcal{T}(0,0)\Delta = \mathcal{S}\Delta + DN(0)\Delta = \mathcal{S}\Delta.$$

$\square$

**Lemma 4.2.** *The operator $\mathcal{S}$ acting on $H^n_{\geq 2}$ and defined by*

$$(\mathcal{S}\Delta)(s) = A\Delta(s) - \Delta(\lambda s)$$

*is boundly invertible in $H^n_{\geq 2}$, i.e, $\forall \eta \in H^n_{\geq 2}$ $\exists! \Delta \in H^n_{\geq 2}$ such that $\mathcal{S}\Delta = \eta$.*

*Proof.* Given $\eta \in H^n_{\geq 2}$ with $\eta(x) = \sum_{n=2}^\infty \eta_n x^n$, we look for $\Delta(x) = \sum_{n=2}^\infty \Delta_n x^n$ such that

$$A\Delta(x) - \Delta(\lambda x) = \eta(x)$$
$$\eta(x) = \sum_{n=2}^\infty \eta_n x^n = \sum_{n=2}^\infty (A - \lambda^n Id)\Delta_n x^n$$
$$\text{Hence,} \quad \Delta_n = (A - \lambda^n Id)^{-1}\eta_n \quad \text{n} \geq 2$$

$\Delta_n$ is well defined because of the third condition and $\|\Delta\| < \infty$ due to $\|\lambda\| < 1$,

$$\Delta(s) = \sum_{n\geq 2} (A - \lambda^n)^{-1}\eta_n s^n$$

where $\left\|(A - \lambda^n)^{-1}\right\| \leq C$ due to $\lambda^n \to 0$. Hence, $\mathcal{S}$ is invertible and $\|\mathcal{S}\| \leq C$.

$\square$

By Proposition 4.1, Lemma 4.2 and applying the Implicit Function Theorem in Banach spaces, exist U an neighborhood of the origin and an analytic map $K^{>}$

$$\begin{aligned} K^{>} : &U \longrightarrow H^{d}_{\geq 2} \\ &K_1 \longrightarrow K^{>}[K_1] \end{aligned}$$

such that $\mathcal{T}(K_1, K^{>}[K_1]) = 0$.
Then, the image of K = $K_1 + K^{>}$ is the invariant manifold claimed in Theoreme 1.1 on the condition tha we choose $K_1$ is parallel $v$ with $\|K_1\|$ small enough.
On the condition that $K(t)$ satisfies 4.1 and $\beta \in \mathbb{C}$, then $K(\beta t)$ also satisfies 4.1. Moreover, when t = 0, $K'(0) = \beta K'(0) = \beta K_1$.Applying the remark 2.35 of the implicit function theorem, we obtain $\widehat{K}(t) = K(\beta t)$ for t small enough and $\widehat{K}$ is stated in 1.1.

### 4.1.2 Non-resonant invariant manifolds for maps

We are going to prove the theorem 1.3 in the real-analytic case. We want to find K of form

$$K = K^{\leq} + K^{>}$$

where $K^{\leq}(s) = \sum_{i=1}^{L} K_i(s)$ and $R(s) = \sum_{i=1}^{L} R_i(s)$ are polynomials of degree L. Keep in mind that $K_i$ is a symmetric i-linear operator in $E^{\otimes i}$ evaluate in $\mathbb{R}^n$ and R is a symmetric i-linear operator in $E^{\otimes i}$ taking values in E.

The strategy to prove Theorem 1.3 is similar to Theorem 1.1. First, we find the $K^{\leq}$ and $R$ matching power of $s$ under appropriate non-resonance conditions. Then, write down the functional equation for case multi-dimensional maps in Banach spaces. Finally, searching for $K^{>}$ to solve the equation that leads to an invariant parameterization.

Before proving the theorem 1.3, we will state some properties that later will be quiet useful. Firstly, we state a general result in linear algebra.

**Proposition 4.3.** *Let $L : E \longrightarrow E$ linear map such that $E = KerL \oplus ImL$. Then, for all $v \in$ Im L exist a unique $u \in$ Im L such that Lu =v.*

*Proof.* Fisrt, we prove the existence of $u$. If $v \in$ Im L, that means existing $u_0 \in E$.In particular, exist a unique $k \in$ Ker L, $u \in$ Im L such that $u_0 = k + u$. Note that

$$v = Lu_0 = Lk + Lu = Lu$$

Then, the uniqueness of u. Suppose $u_1, u_2 \in$ Im L such that $Lu_1 = v, Lu_2 = v$, therefore

$$0 = Lu_1 - Lu_2 = L(u_1 - u_2) \quad \text{so,} \quad u_1 - u_2 \in \text{Ker L}$$

$u_1 \in$ Ker L + $u_2$ where $u_2 \in ImL$, at same moment $u_1 \in ImL$. Using the uniqueness of the descomposition, we get $u_1 = u_2$. □

**Complexification and Realification**

Even though the original vector field is real, we use the trick Complexification to transform in complex.Because to transform in a diagonal matrix is easier to deal with complex, that is the reason to introduce the complexification and the opposite realification.

Let P be a matrix which columns are real vectors or come in pairs of complex conjungate vactors correspond to eigevalues of real or complex conjugate eigenvalues of the matrix $DF(0)$, respectively.

Firstly, We will state a methodology to get a complex parameterization $K^{\mathbb{C}}$ of a real manifold and the corresponding dynamics $R^{\mathbb{C}}$ in the following way.

If we detote Q is the permutation and idempotent matrix which permutes the complex conjugate columns of P, then

$$PQ = \overline{P}, \quad Q\Lambda Q = \bar{\Lambda}$$

In addition, if we denote $Q_L$ the permutation matrix correspoding to the matrix L which forms by just taking the first j columns of P.

$$LQ_L = \bar{L}, \quad Q_L \Lambda Q_L = \bar{\Lambda}$$

Using the change of variable above,we obtain $K^{\mathbb{C}}$ and $R^{\mathbb{C}}$.

Now, we do the oppsite ,i.e, realification.Note that the permutation matrices of order two are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \widetilde{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, one permuting block of $Q$, denoting $Q_k$is either 1 or the matriz $\widetilde{I}$ and we associate

$$C_k = \begin{cases} 1 & \text{if} \quad Q_k = 1 \\ \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} & \text{if} \quad Q_k = \widetilde{I} \end{cases}$$

Note that for any permutation block $Q_k = C_k \bar{C}_k^{-1}$, where $\bar{C}_k^{-1}$ is denoted the inverse of the conjugate matrix $C_k$. Then, we obtain $C_L$ by collecting $C_k$ matrices corresponding to L. Finally, we obtain the reaL parameterization $K(s) = K^{\mathbb{C}}(C_L s)$ and the corresponding real dynamics is $R(s) = C_L^{-1} R^{\mathbb{C}}(C_L s)$.

$$K(\bar{s}) = \overline{K(s)} \quad \text{and} \quad R(\bar{s}) = \overline{R(s)}$$

From now on, we will consider that the parameterizations of real invariant manifolds are real. Even we have used complex parameteriazations(and get a complex power series in the intermediate step).

**Definition 4.4.** *Let X, Y be vector spaces and assume $\mathcal{L}^i_{A,B}$ on the the space of symmetric i-linear operators from X to Y of form*

$$\begin{aligned} \mathcal{L}^i_{A,B} : & L^i_s(X;Y) \longrightarrow L^i_s(X;Y) \\ & M \qquad \longrightarrow \mathcal{L}^i_{A,B} := AM - MB^{\otimes i} \end{aligned}$$

*where $A : Y \to Y, B : X \to X$ are linear maps. For each M, $\mathcal{L}^i_{A,B}M$ is said to be Sylvester operators.*

**Proposition 4.5.**

$$\begin{aligned} Spec(\mathcal{L}^i_{A,B}) &= Spec(A) - (Spec(B))^i \\ &= \{\lambda - \mu_1 \cdots \mu_i \mid \lambda \in Spec(A), \mu_1, \cdots \mu_i \in Spec(B)\}. \end{aligned} \tag{4.7}$$

*Proof.* Since the left- and right-hand side of 4.7 are continuous respect to the matrices A of dimension $\ell$ and B of dimension $k$, it is sufficient to prove for a dense set f matrices, i.e, when A and B are diagonalizable over the complex.Using a change of basis we could assume that the matrices A and B are diagonals matrices. Indeed,we consider that A,B are diagonal matrices. In addition, $\{\lambda_j\}_{1\leq j\leq \ell}$ and $\{\mu_j\}_{1\leq j\leq k}$ are the eigenvalues of A and B respectively.

Let $M$ be a symmetric i-linear operator from X to Y.

$$\begin{aligned}
\mathcal{L}^i_{A,B}M(s) &= AM(s) - MB(s) = AM(s) - M(\mu_1 s_1, \cdots, \mu_k s_k) \\
&\overset{(a)}{=} \begin{pmatrix} \lambda_1 M^1(s) - M^1(\mu_1 s_1, \cdots, \mu_k s_k) \\ \vdots \\ \lambda_1 M^\ell(s) - M^\ell(\mu_1 s_1, \cdots, \mu_k s_k) \end{pmatrix} \\
&\overset{(b)}{=} \begin{pmatrix} \lambda_1 \sum_{|n|=i} M^1_n s^n - \sum_{|n|=i} M^1_n \mu_1^{n_{1_1}} s_1^{n_{1_1}} \cdots \mu_k^{n_{k_1}} s_k^{n_{k_1}} \\ \vdots \\ \lambda_\ell \sum_{|n|=i} M^\ell_n s^n - \sum_{|n|=i} M^\ell_n \mu_j^{n_{\ell_1}} s_1^{n_{\ell_1}} \cdots \mu_k^{n_{\ell_k}} s_k^{n_{\ell_k}} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{|n|=i} (\lambda_1 - \mu_1^{n_{1_1}} \cdots \mu_k^{n_{1_k}}) M^1_n s^n \\ \vdots \\ \sum_{|n|=i} (\lambda_\ell - \mu_1^{n_{\ell_1}} \cdots \mu_k^{n_{\ell_k}}) M^\ell_n s^n \end{pmatrix}
\end{aligned}$$

(a) $M(s)$ $\in$Y so we can write as $(M^1(s), \cdots, M^\ell(s))$ due to Y is a vector space, analoguous way to $M(\mu_1 s_1, \cdots, \mu_k s_k)$.

(b) Using the definition of 2.19 $M(s)$ is a homogeneous polynomial of m degree in k variables, then using multi-index notation $M^i(s) = \sum_{|n|=i} M^i_n s^n$ for all $1 \leq i \leq j$. Hence, $\mathcal{L}^k_{A,B}$ are defined on $\mathbb{C}^k_k[s]$.

Note that the set $\{\{s_1^{n1}, \cdots, s_k^{n_k}\} \mid n_1 + \cdots + n_k = i\}$ is the standard basis of $\mathbb{C}^k_i[s]$ and denote

$$N := \binom{i+k-1}{k-1}$$

where N stands for the number of elements on $\mathbb{C}^k_i[s]$.

If we rewrite $\mathcal{L}^i_{A,B}$,which is the i-throw of $\mathcal{L}_{A,B}$, in the standard basis of $\mathbb{C}^k_i[s]$.we obtain

$$\widetilde{M} = \begin{pmatrix} \lambda_1 - \mu_1^{n_{1_1}} \cdots \mu_k^{n_{1_k}} & 0 & \cdots & 0 \\ 0 & \lambda_1 - \mu_1^{n_{2_1}} \cdots \mu_k^{n_{2_k}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 - \mu_1^{n_{N_1}} \cdots \mu_k^{n_{N_k}} \end{pmatrix}$$

where $\widetilde{M}$ is a matrix of N $\times$ N. Hence,$\mathcal{L}^i_{A,B}$ is diagonalizable and so is $\mathcal{L}_{A,B}$. In addition, 4.7 holds due to $\widetilde{M}$. $\square$

**Lemma 4.6.** *Assume that $(Spec(A_E))^i \cap Spec(A_C) = \emptyset$ for $i = 2, \cdots, L$ and $r \geq L$. Then, we can find polynomials $K^{\leq}, R$ as before in such a way that*

$$D^j(F \circ K^{\leq} - K^{\leq} \circ R)(0) = 0 \tag{4.8}$$
$$K^{\leq}(0) = 0, \quad DK^{\leq}(0) = (Id, 0) \tag{4.9}$$
$$R(0) = 0, \quad DR(0) = A_E \tag{4.10}$$

*Moreover, if we assume that N = F - A is sufficiently small, then $K^{\leq}$ - (Id,0) and R -$A_E$ will be arbitrarily small.*

*Proof.* Firstly, for the case $j = 0$ is necessary satisfy $K^{\leq}(0) = 0$ and $R(0) = 0$. Therefore, we take $K_0 = 0, R_0 = 0$. For the case j = 1 is needed to hold $AK_1 = K_1 R_1$, then we pick $R_1 = A_E$ and $K_1 = I_x$. Taking projections over the spaces E and C denoting by $K_j^x = \Pi_E K_j$ and $K_j^y = \Pi_C K_j$

$$K(s) = \begin{pmatrix} s \\ 0 \end{pmatrix} + \sum_{m \geq 2} \begin{pmatrix} K_m^x(s) \\ K_m^y(s) \end{pmatrix}$$
$$R(s) = A_E s + \sum_{m \geq 2} R_m(s)$$

to satisfy 4.9 and 4.10. Then, assuming that the terms up to m-1 are known as before

$$F(K(s)) - K(R(s)) = [F(K_{<m}(s))]_{\leq m} + DF(0)K_m(s) - [K_{<}m(R_{<m})(s)]_{\leq m}$$
$$- DK(0)R_m(s) - K_m(A_E s) + \cdots$$

Equating the term m

$$AK_m(s) - \begin{pmatrix} Id \\ 0 \end{pmatrix} R_m(s) - K_m(A_E s) = C_m(s) \tag{4.11}$$

where $C_m(s) = [K_{<m}(R_m(s))]_m - [F(K_{<m}(s))]_m$ is a homogeneuous polynomial and depend on $\{K_1, \cdots, K_{m-1}, R_1, \cdots, R_{m-1}\}$.Furthermore, $C_m(s)$ is known.
we reformulate 4.11

$$\begin{pmatrix} A_E & B \\ 0 & A_C \end{pmatrix} \begin{pmatrix} K_m^x(s) \\ K_m^y(s) \end{pmatrix} - \begin{pmatrix} R_m(s) \\ 0 \end{pmatrix} - \begin{pmatrix} K_m^x(A_E s) \\ K_m^y(A_E s) \end{pmatrix} = \begin{pmatrix} C_m^x(s) \\ C_m^y(s) \end{pmatrix}$$

Finally, we obtain two equation for $K_m$

$$\begin{aligned} A_E K_m^x(s) + BK_m^y(s) - R_m^x(s) - K_m^x(A_E s) &= C_m^x \\ A_C K_m^y(s) - K_m^y(A_E s) &= C_m^y \end{aligned} \tag{4.12}$$

Using the operator $\mathcal{L}$, we can rewrite 4.12

$$\mathcal{L}_{A_E, A_E}^m K_m^x(s) = C_m^x(s) + R_m(s) - BK_m^y(s) \tag{4.13}$$
$$\mathcal{L}_{A_C, A_E} K_m^y(S) = C_m^y(s) \tag{4.14}$$

Since we proved Proposition 4.7 and by hipothesis (iii) of the theorem 1.3, $\mathcal{L}_{A_C, A_E}^m$ is invertible. Since $C_m^y(s)$ is known, $K_m^y(s)$ is obtained by $(\mathcal{L}_{A_C, A_E})^{-1} C_m^y(s)$. Note that

$$\mathbb{C}_m^k[s] = Ker \mathcal{L}_{A_E, A_E}^m \oplus Im(\mathcal{L}_{A_E, A_E}^m) \tag{4.15}$$

Once we have solved (4.14), we have to solve (4.13). Notice that $\widehat{C}_m^x := C_m^x(s) - BK_m^y(s)$ is known, we can get $K_m^x(s)$ on condition that we choose the terms $R_j = 0$

$K_m^x$ will be obtained as a consequence of Proposition 4.3 applied to (4.15). We can descompose $\widehat{C}_m^x = \widehat{N}_m^x + \widehat{Im}_m^x \in Ker\mathcal{L}^m_{A_E,A_E} \oplus Im(\mathcal{L}^m_{A_E,A_E})$. Finally, we choose $R_m(s) = -\widehat{N}_m^x$ and $K_m^x$ as the unique solution of

$$\mathcal{L}^m_{A_E,A_E} K_m^x(s) = \widehat{Im}_m^x$$

Using 4.15 and Proposition 4.3, we can solve 4.13.

Notice that since the eigenvalues of $A_e$ are of modulus smaller than 1, then there is L such that Ker $\mathcal{L}^m_{A_E,A_E}$ is 0 for $m \geq L$ . Therefore R is a polynomial .

The last statement of the lemma follows observing that the coefficients of $K^{\leq} - (Id, 0), (R - A_E)$ are algebraic expressions of the coefficients of N = F - A. $\square$

*Proof.* of Theorem 1.3

We start by taking an adapted norm in $\mathbb{R}^n$ such that

$$\|A_E\| < \rho(A_E) + \epsilon$$
$$\left\|A^{-1}\right\| < \rho(A^{-1}) + \epsilon \quad \text{for some } \epsilon \text{ small enough}$$

where $\rho$ stands for the spectral radius.

Then, using hypothesis (iv) we get $\left\|A^{-1}\right\| \cdot \|A_E\|^{L+1} < 1$. We can rewrite $F \circ K = K \circ R$ as a functional equation

$$\begin{aligned}\mathcal{T}(N, K^{>})(s) =& A(K^{\leq}[N](s) + K^{>}(s)) + N(K^{\leq}[N] + K^{>}(s)) - K^{\leq}[N](s) \circ R[N](s) \\ & - K^{>} R[N](s) = 0.\end{aligned} \tag{4.16}$$

where we emphasize that $K^{\leq}[N](s), R[N](s)$ are depen on N as we stated in the proof of the Lemma 4.6.

Firstly, we suppose N small enough to prove Theorem 1.3, then using a scalling technique the Theorem 1.3 will be proved.

We define

$$H^{d,n}_{\delta,k} = \{G : \overline{B}_\delta(0) \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n \mid G = \sum_{i=k}^{\infty} g_i(s), \sum_{i=k}^{\infty} |g_i| \, \delta^i < \infty\}$$

where $g_i(s)$ is a homogeneuous polynomial evaluate in $s^{\otimes i}$ of form $\sum_{|m|=i} g_{i,m} s^m$ and its norm $|g_i| = \sum_{i=k}^{\infty}$. In addition, $H^{d,n}_{\delta,k}$ invested with the norm $\|G\|_B = \sum_{i=k}^{\infty} |G_i| \, \delta^i$.

Considering $\mathcal{T} : H^{n,n}_{3,2} \times H^{d,n}_{2,L+1} \longrightarrow H^{d,n}_{2,L+1}$ Moreover, if N = 0 then $K^{\leq} = I_E$ and $R = A_E$. Hence, $\mathcal{T}(0, 0) = 0$. $\square$

We will state a Proposition that ensures $\mathcal{T}$ is analytic and a Lemma $D_2\mathcal{T}(0, 0)$ is invertible in order to apply The Implicit Function theorem.

**Proposition 4.7.** *We have*

*(i) The operator* $\mathcal{T} : V \subset H^d_{3,2} \times H^d_{2,L+1} \longrightarrow H^d_{2,L+1}$ *is* $C^\infty$ *in a neighborhood V of (0,0).*

*(ii)* $D_2\mathcal{T}(0, 0) = \mathcal{S} = A\Delta - \Delta \circ A_E$.

*Proof.* The proof is proved by using the same strategy as Proposition 4.1. □

**Lemma 4.8.** *$\mathcal{S}$ from $H^{d,n}_{2,L+1}$ to $H^{d,n}_{2,L+1}$ is boundly invertible.*

*Proof.* We use the same strategy as the case one-dimensional.Considring

$$A\Delta - \Delta \circ A_E = \eta$$

where $\eta = \sum_{i=L+1}^{\infty} \eta_i(s)$ and $\Delta = \sum_{i=L+1}^{\infty} \Delta_i(s)$ matching the power, we obtain

$$A\Delta_i - \Delta_i(A_E(s)) = \eta_i, \quad i \geq L+1$$
$$\text{Then,} \quad \Delta_i = A^{-1}\Delta_i(A_E(s)) + A^{-1}\eta_i(s) \quad i \geq L+1$$

By hipothesis (ii) and ( $\left\|A^{-1}\right\| \cdot \|A_E\|^{L+1}$ < 1), we have $\left\|A^{-1}\right\| \cdot \|A_E\|^{i}$ < 1 , for $i \geq L+1$

$$\|\Delta_i\| \leq \left\|A^{-1}\right\| \|\Delta_i\| \|A_E\|^i + \left\|A^{-1}\right\| \|\eta_i\|$$

Then,

$$\|\Delta_i\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|A_E\|^i} |\eta_i| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|A_E\|^{L+1}} |\eta_i|$$

Hence,

$$\sum_{i \geq L+1} |\Delta_i| \, 2^i \leq C \cdot \sum_{i \geq L+1} |\eta_i| \, 2^i < \infty$$

□

The hypothesis of the Implicit Function theorem for $\mathcal{T}(N, K^>) = 0$ are given by Proposition 4.7 and Lemma 4.8. Therefore,$\mathcal{T}(N, K^>[N]) = 0$ for N small enough.

In fact, the Theorem 1.3 is true for a general N using a scalling technique. Observe that if we consider $F^\delta(s) = \frac{1}{\delta}F(\delta s)$ for a small enough $\delta$, then $N^\delta$ is small. As a consequence, we obtain $K$ and $R$ for

$$F^\delta \circ K = K \circ R \tag{4.17}$$

Considering $K^{1/\delta}(s) = \delta K(\frac{1}{\delta}s), R^{1/\delta}(s) = \delta R(\frac{1}{\delta}s)$, we use a change of variable $x = \frac{s}{\delta}$ and the preceding notations in 4.17. We have $K, R$ for a general N

$$F \circ K^{1/\delta} = K^{1/\delta} \circ R^{1/\delta}$$

in a neighborhood of 0.

## 4.2 The parameterization method for flows

In this section we will prove the parameterization method for flows that we stated in the chapater 1. The main idea of the proof is using the same strategy as the case for maps.Looking for maps $K$ and $R$, by matching the powers we can find $K_m$ for m>1, then we can find $K^{\leq} = \sum_{m=0}^{L} K_m(s) = \sum_{m=0}^{L} K_m s^m$ which is a polynomial of degree L. We write K of form

$$K = K^{\leq} + K^{>}$$

where $K^>$ is a fucntion vanishing at the origein and the first L derivatives.Then, write a functional eqaution for $K^>$ and the non-linear part of $\mathcal{X}$, N and appliying the Implicit Function Theorem in appropriat framework.

### 4.2.1 Non-resonant invariant manifolds for flows

Analoguos to the proof of 1.3, we will define the operator $Spec(\widetilde{\mathcal{L}}^i_{A,B})$.But their spectrum is different

**Definition 4.9.** *Let X, Y be vector spaces and assume $Spec(\widetilde{\mathcal{L}}^i_{A,B})$ on the the space of symmetric i-linear operators from X to Y of form*

$$\begin{aligned} \widetilde{\mathcal{L}}^i_{A,B} : & L^i_s(X;Y) \longrightarrow L^i_s(X;Y) \\ & K \qquad \longrightarrow \widetilde{\mathcal{L}}^i_{A,B}K := AK - DK \cdot B^{\otimes i} \end{aligned}$$

*where $A : Y \to Y, B : X \to X$ are linear maps.*

**Proposition 4.10.**

$$\begin{aligned} Spec(\widetilde{\mathcal{L}}^i_{A,B}) &= Spec(A) - iSpec(B) \\ &:= \{\lambda - (\mu_1 + \cdots + \mu_i) \mid \lambda \in Spec(A), \mu_1, \cdots, \mu_i \in Spec(b)\}. \end{aligned} \tag{4.18}$$

*Proof.* Following the same strategy to the proposition 4.5. Due to the continuity of the objects in 4.18 respect to the matrices A and B. It suffices to prove 4.18 for the dense subset of diagonalizable matrices A and B. We use the complexification and realification as we stated in case for maps in order to transform the diagonalizable matrices A and B in diagonal matrices. Hence, we consider A and B are diagonal matrices of dimension $\ell$ and k respectively. Let $\{\lambda_1, \cdots \lambda_\ell\}, \{\mu_1, \cdots, \mu_k\}$.

Given K a symmetric $i$-linear operator, by definition $K(s)$ is a homogeneous polynomial of degree $i$.

$$\begin{aligned} \widetilde{\mathcal{L}}^i_{A,B} &= \begin{pmatrix} \lambda_1 K^1(s) \\ \vdots \\ \lambda_\ell K^\ell(s) \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial s_1}K^1(s) & \cdots & \frac{\partial}{\partial s_k}K^1(s) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial s_1}K^\ell(s) & \cdots & \frac{\partial}{\partial s_k}K^\ell(s) \end{pmatrix} \begin{pmatrix} \mu_1 s_1 \\ \vdots \\ \mu_k s_k \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 K^1(s) - \sum_{j=1}^i \frac{\partial}{\partial s_j} K^1(s)\mu_j s_j \\ \vdots \\ \lambda_\ell K^\ell(s) - \sum_{j=1}^i \frac{\partial}{\partial s_j} K^\ell(s)\mu_j s_j \end{pmatrix} \\ &= \begin{pmatrix} \sum_{|m|=i} \lambda_1 K^1_m - \sum_{j=1} \left( \sum_{|m|=i} n_j K^1_m s^{n_1} \cdots s^{n_j-1} \cdots s_k^{n_k} \right) \\ \vdots \\ \sum_{|m|=i} \lambda_\ell K^\ell_m - \sum_{j=1} \left( \sum_{|m|=i} n_j K^\ell_m s^{n_1} \cdots s^{n_j-1} \cdots s_k^{n_k} \right) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{|m|=i} \left( \lambda_1 - \sum_{j=1}^i n_j \mu_j \right) K^1_m s^m \\ \vdots \\ \sum_{|m|=i} \left( \lambda_\ell - \sum_{j=1}^i n_j \mu_j \right) K^\ell_m s^m \end{pmatrix} \end{aligned}$$

The last equality is hold by the permutation of the finite summations.As the previous section, using the standard basis of $\mathbb{C}^k_i[s]$ which has N=$\binom{k+i-1}{k-1}$ elements. Then, if we write each row of

$\widetilde{\mathcal{L}}^i_{A,B}$ in $\mathbb{C}^k_i[s]$, we obtain for each r-th row

$$M = \begin{pmatrix} \lambda_r - \sum_{|m|=i} n_j \mu_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N - \sum_{|m|=i} n_j \mu_j \end{pmatrix} \in M_{N\times N}$$

Hence, each row of $\widetilde{\mathcal{L}}^i_{A,B}$ is diagonalizable matrix in $\mathbb{C}^k_i[s]$. In particular their eigenvalues are the form $\lambda_r - (\mu_1 + \cdots + \mu_i)$, then 4.18 holds. □

**Lemma 4.11.** *Given $\mathcal{X} : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n, \mathcal{X}(0) = 0, \mathcal{X} \in \mathcal{C}^L$, satisfying hypothesis (iii) of Theorem 1.3, we can find polynomials $K^{\leq}$ and R of degree not bigger than L such that*

$$D^j(\mathcal{X} \circ K^{\leq} - DK^{\leq} \cdot R)(0) = 0, \quad 0 \leq j \leq L, \tag{4.19}$$

$$K^{\leq}(0) = 0, \tag{4.20}$$

$$R(0) = 0, DR(0) = A_E \tag{4.21}$$

*Moreover, if we assume that $N = F - A$ and B are sufficiently small, then $K^{\leq} - (Id, 0)$*

*Proof.* We look for $K^{\leq}(s) = \sum_{m=0}^{L} K_m s^m, R(s) = \sum_{m=0}^{L} R_m(s)$ so as to satisfy $\mathcal{X} \circ K = DK \cdot R$ up to order L. We denote $\Pi_E$ and $\Pi_C$ the projections onto E and C, then $K^E_m = \Pi_E K_m,\ K^C_m = \Pi_C K_m$. Taking $K_0 = 0,\ K^E_1 = Id,\ K^C_1 = 0,\ R_0 = 0,\ R_1 = A_E$ satisfy 4.19. Then,

$$K^{\leq}(s) = \begin{pmatrix} s \\ 0 \end{pmatrix} + \sum_{m=1}^{L} K_m s^m, \quad R(s) = A_E s + \sum_{m=1}^{L} R_m s^m$$

Finding the rest of the term by matching the powers.Assuming the term up to m of $K^{\leq}$ and $R^{\leq}$ are known, then we want to find the term m

$$\begin{aligned} \mathcal{X}(K(s)) - DK(s) \cdot R(s) &= \mathcal{X}(K^{<m}(s) + K_m(s) \cdots) \\ &- (DK^{<m}(s) + DK_m(s) + \cdots)(R^{<m}(s) + R_m(s) + \cdots) \\ &= [\mathcal{X}(K^{<m}(s))]_m + D\mathcal{X}(0)K_m(s) + \cdots - [DK^{<m}(s) \cdot R^{<m}(s)]_m \\ &- DK_m(s) A_E s - \begin{pmatrix} Id \\ 0 \end{pmatrix} R_m(s) + \cdots \end{aligned}$$

the m-th term of the above equation

$$AK_m(s) - DK_m(s) A_E s - \begin{pmatrix} Id \\ 0 \end{pmatrix} R_m = C_m(s)$$

where $C_m(s) = -[\mathcal{X}(K^{<m}(s))]_m + [DK^{<m} R]_m$ are known. Using the projection notations that we introduced before

$$\begin{pmatrix} A_E & B \\ 0 & A_C \end{pmatrix} \begin{pmatrix} K^E_m(s) \\ K^C_m(s) \end{pmatrix} - DK_m(s) A_E s - \begin{pmatrix} R_m \\ 0 \end{pmatrix} = \begin{pmatrix} C^E_m \\ C^C_m \end{pmatrix}$$

is same to

$$\begin{aligned} &A_E K^E_m(s) + BK^C_m(s) - DK^E_m(s) A_E s - R_m(s) = C^E_m(s) \\ &A_C K^C_m(s) - DK^C_m(s) A_E s = C^C_m(s) \end{aligned}$$

Using the operator that we defined in the preceding proposition in above equation

$$\widetilde{\mathcal{L}}^m_{A_E,A_E} K^E_m = -BK^C_m + C^E_m + R_m \tag{4.22}$$

$$\widetilde{\mathcal{L}}^m_{A_C,A_E} K^C_m = C^C_m(s) \tag{4.23}$$

The solution of 4.22 and 4.23 is found the same way to the case for maps and it is found by the following procedure.

(i) Solve equation 4.22
Note that, $\lambda - (\mu_1 + \cdots + \mu_i) \neq 0$ holds for $\lambda \in Spec(A)$ and $\mu_1, \cdots, \mu_i \in Spec(A_E)$, then $\widetilde{\mathcal{L}}^i_{A_C,A_E}$ invertible.Hence, this equation has solution and is unique due to the statement (iv) of theorem 1.3..

(ii) Choose $R_m$ so that $\widetilde{C}^E_m := C^E_m - BK^C_m$ belongs to the range of $\widetilde{\mathcal{L}}^m_{A_E,A_E}$.

(iii) solve equation 4.23 so as to fin $K^E_m$

Since we can write $\widetilde{C}^E_m(s) = \widetilde{N}^E_m(s) \oplus \widehat{I}^E_m(s)$ the unique descomposition where $\widetilde{N}^E_m(s) \in \text{Ker}\, \widetilde{\mathcal{L}}$ $\widehat{I}^E_m(s) \in Im\widetilde{\mathcal{L}}$. Using the general result of linear algebra 4.3 and choosing $R_m(s) = -\widetilde{N}_m(s)$.We get $K^E_m$ such that $\widetilde{\mathcal{L}}^m_{A_E,A_E}$.

The last statement in Lemma 4.11 holds because $K^{\leq} - (Id, 0)$ and $R - A_E$ are obtained by using a finite number of algebraic calculations of coefficients of N. ☐

We consider $\mathcal{X} = A + N$ then the equation $\mathcal{X} \circ K = DK \cdot R$ can be writen as

$$U(N, k^{>}) := AK^{\leq}[N] - DK^{\leq}[N] \cdot R + AK^{>} + N(K^{\leq}[N] + K^{>}) - DK^{>} \cdot R[N] = 0$$

$K^{\leq}[N], R[N]$ stand for dependence of N.

We wiil work with the space of analytic functions

$$H^{d,n}_{\delta,k} = \{F : \overline{B}_\delta(0) \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n \mid F(s) = \sum_{m=k}^{\infty} F_m s^m, \sum_{m=k}^{\infty} \|F_m\| \delta^j < \infty\} \tag{4.24}$$

endowed with the norm $\|F\| := \sum_{m=k}^{\infty} \|F_m\| \delta^j$.

**Proposition 4.12.** *If N is analytic, then:*

*(i) The operator $\mathcal{U} : V \subset H^{n,n}_{3,2} \times H^{d,n}_{2,L+1} \longrightarrow H^{d,n}_{2,L+1}$ is analytic in a neighborhood V of N= 0,$K^{>} = 0$.*

*(ii) $D_2U(0,0)\Delta = A\Delta - (D\Delta) \cdot A_E$.*

*Proof.* As we said before $K^{\leq}$ and $R$ depend analytically on N. Using the scalling thecnique we can suppose N is as small as we want. Then, $K^{\leq}$ and $R$ will be as close as we want to the immersion of E $\subset \mathbb{R}^d$ and $A_E$, respectively. Therefor $U$ is well defined. Finally, applying Omega-Lemma that assure the composition of analytic functions is analytic.

Moreover,

$$\begin{aligned} U(0, \Delta^{>}) &= U(0,0) + D_2U(0,0)\Delta + o(\|\Delta\|) \\ &= AK^{\leq} - DK^{>} \cdot R + D_2U(0,0)\Delta + o(\|\Delta\|) \end{aligned}$$

On the other hand, $U(0, \Delta^{>}) = AK^{\leq}[0] + A\Delta^{>} - DK^{\leq} \cdot R - D\Delta^{>}R[0]$

Hence, the statement (ii) is hold

$$D_2U(0,0)\Delta = -D\Delta \cdot R + A\Delta + o(\|\Delta\|)$$

$\square$

Let us introduce some aspects will be needed later, consider

$$\mu_+ = \sup\{Re\mu \mid \mu \in Spec(A_E)\}$$
$$\lambda_- = \inf\{Re\lambda \mid \lambda \in Spec(A)\}$$

Using the hypothesis (ii)and the hypothesis (iv) of the theorem 1.5, there exist $\epsilon > 0$ small enough such that

$$\mu_+ + \epsilon < 0$$
$$-\lambda_- + (L+1)\mu_+ + (L+2)\epsilon < 0$$

we take a adapted norm in $\mathbb{C}^d$ such that

$$\left|e^{A_E t}y\right| \leq e^{(\mu_+ + \epsilon)t}\,|y| \quad t \geq 0 \text{ for all } y \in \mathbb{R}^d \tag{4.25}$$

and such that $\|B\|$ is as small as we need. Before we prove the theorem, we shall introduce some notations.

**Lemma 4.13.** *The operator*

$$\mathcal{S}\Delta := D_2U(0,0)\Delta = A\Delta - (D\Delta)\cdot A_E$$

*is boundedly invertible fromm $H^{d,n}_{2,L+1}$ to itself.*

*Proof.* $\mathcal{X}$ is invertible provided that given $\eta \in H^{d,n}_{2,L+1}$ there exists a unique $\Delta \in H^{d,n}_{2,L+1}$ such that $\mathcal{X}\Delta = \eta$, that is ,

$$D\Delta(s)A_E s - A\Delta(s) = -\eta(s) \tag{4.26}$$

We will use different method to case for maps.However, the preceding strategy is valid tanslate to this case. It gives an explicit formulation for the solution.

Using a change of variable $s = e^{A_E t}y$ in $\Delta(s)$

$$\widetilde{\Delta}(t,y) = \Delta(e^{A_E t}y) \tag{4.27}$$

By (4.25) $\widetilde{\Delta}$ is well defined for $|y| \leq 2$ and $t \geq 0$. Equation 4.26 becomes

$$\frac{d}{dt}\widetilde{\Delta}(t,y) - A\widetilde{\Delta}(t,y) = \eta(e^{A_E t}y) \tag{4.28}$$

Note that equation 4.28 is a linear and non-homogeneous ordinary differential eqaution. Applying the so-called variation of constants formula (in paricular stated in [Sot79]). We get

$$\widetilde{\Delta}(t,y) = e^{At}\left[e^{-At_0}\widetilde{\Delta}(t_0,y)\text{- !!! }\int_{t_0}^{t} e^{-A\tau}\eta(e^{A_E\tau})d\tau\right] \quad t_0, t \in [0,\infty) \tag{4.29}$$

where $e^{At}$ is the funfamental matriz of $\frac{d}{dt}\widetilde{\Delta}(t,y) = A\widetilde{\Delta}(t,y)$.

The heuristic idea is to take $t = 0$ in 4.29 and letting $t_0 \to \infty$.Hence, we consider

$$\Delta(y) = \widetilde{\Delta}(0, y) = \int_0^\infty e^{-A\tau}\eta(e^{A_E\tau}y)d\tau \tag{4.30}$$

Since $\eta \in H^{d,n}_{2,L+1}$, we put $\eta = \sum_{j=L+1}^\infty \eta_j(s)$ in 4.30.

$$\Delta(y) = \int_0^\infty e^{-A\tau} \sum_{j=L+1}^\infty \eta_j(e^{A_E\tau}y)d\tau$$

Now, let us see $\Delta$ we difine above belong to $H^{d,n}_{2,L+1}$.

$$\begin{aligned}\left\| \int_0^\infty e^{-A\tau} \sum_{j=L+1}^\infty \eta(e^{A_E}y)d\tau \right\| &\leq \int_0^\infty \left\| e^{-A\tau} \right\| \cdot \sum_{j=L+1}^\infty \|n_j\| \cdot \left\| e^{A_E\tau} \right\|^j d\tau \\ &\leq \sum_{j=l+1}^\infty \left\| e^{-A\tau} \right\| \cdot \|n_j\| \cdot \left\| e^{A_E\tau} \right\|^j d\tau\end{aligned}$$

The last inequality is hold due to $\sum_{j=L+1}^\infty \|n_j\| \cdot \|e^{A_E\tau}\|^j < \infty$ for $|y| \leq 2, t \geq 0$. There exits M due to $|e^{-At}|\, e^{(\lambda-\epsilon)t} \underset{t\to\infty}{\longrightarrow} 0$ such that $|e^{-At}y| \leq Me^{-(\lambda_- - \epsilon)t}\,|y|$ Since

$$\begin{aligned}\int_0^\infty \left\| e^{-A\tau} \right\| \cdot \left\| e^{A_E\tau}y \right\|^j d\tau &\leq \int_0^\infty Me^{-(\lambda_- - \epsilon)t} e^{j(\mu+\epsilon)t} d\tau \\ &\leq M \int_0^\infty e^{[-\lambda_- + (L+1)\mu_+ + (L+2)\epsilon]t} e^{(j-L-1)(\mu_+ + \epsilon)t} d\tau < \infty\end{aligned}$$

The fact that $[-\lambda_- + (L+1)\mu_+ + (L+2)\epsilon] < 0$ and $(j - L - 1)(\mu_+ + \epsilon) < 0$ due to $j \geq L + 1$ and $\mu_+ + \epsilon < 0$ make the last inequality hold. Therefore

$$\sup_{j \geq L+1} \int_0^\infty \left\| e^{-A\tau} \right\| \cdot \left\| e^{A_E\tau}y \right\|^j d\tau =: C < \infty$$

Finally,

$$\begin{aligned}|\Delta_j(y)| = \left| \int_0^\infty e^{-A\tau}\eta_j(e^{A_E}\tau y)d\tau \right| &\leq \int_0^\infty \left\| e^{-A\tau} \right\| \cdot \|A_E\tau\| \cdot \|\eta_j\| \left| e^{A_E\tau} \right|^j \\ &\leq \|\eta_j\| \cdot |y|^j \int_0^\infty \left\| e^{-A\tau} \right\| \cdot \left\| e^{A_E\tau} \right\|^j d\tau\end{aligned}$$

Hence, $\|\Delta_j\| \leq C\, \|\eta_j\|$ then $\|\Delta\|_{H^{d,n}_{2,L+1}} \leq C\, \|\eta\|_{H^{d,n}_{2,L+1}}$

□

The proof of Theorem 1.5 for N small enough follows from a straightforward application the Implicit Function theorem to Lemma 4.13 and Proposition 4.12 for functional equation $U(N, K^>) = 0$.

Now, we introduce the scaling thecnique in flow version to prove the theorem 1.5 is valid for any N. Consider

$$\mathcal{X}^\delta(x) = 1/\delta\mathcal{X}(\delta x) = Ax + N^\delta(x)$$

for a small enough $\delta$ then $N^\delta$ is small. Applying the Theorem 1.5 for N small enough, we get $K$ and $R$ satisfying

$$F^\delta \circ K = DK \cdot R$$

Hence, $\widetilde{K} = \delta K(x/\delta)$ and $\widetilde{R} = \delta R(x/\delta)$ satisfy

$$\widetilde{\mathcal{X}} \circ \widetilde{K} = D\widetilde{K} \cdot \widetilde{R}$$

# Chapter 5

# Applications

In this section we apply the parameterization method to computethe invariant manifolds of fixed points of two famous models: the Hénon map and the Lorenz system.

## 5.1 Hénon Maps

The Hénon map is a discrete-time dynamical system in the plane and takes points from plane to itself

$$T(x, y) = (y + 1 - ax^2, bx) \tag{5.1}$$

$T$ is invertible and it is

$$T^{-1}(\bar{x}, \bar{y}) = (\frac{1}{b}\bar{y}, \bar{x} - 1 + \frac{a}{b^2}\bar{y}^2)$$

For $a = 1.4, b = 0.3$ is said to be the classical Hénon map. Hénon used numerical investigation to study T in classical case and other values of the parameters. From his numerical computation he conjuctured that there is an attractor that is fractal in nature(a strange attractor).

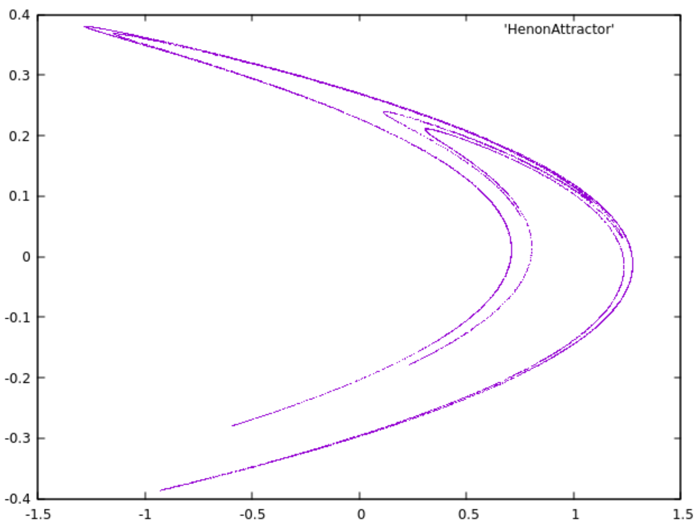



Figure 5.1: Hénon Attractor

### 5.1.1 Stability of fixed points

The fixed points are the solution of $T(x,y) = (x,y)$ and depend on the parameters $a, b$. These sre

$$P_1 = \left(\underbrace{\frac{b-1+\sqrt{(1-b)^2+4a}}{2a}}_{x_1}, bx_1\right), \quad P_2 = \left(\underbrace{\frac{b-1-\sqrt{(1-b)^2+4a}}{2a}}_{x_2}, bx_2\right)$$

Provided that $a > -\frac{(1-b)^2}{4}$.Using the Jacobian matrix of the Hénon map

$$DT(x,y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

where $\det DT(x,y) = -b$ for any points $(x,y) \in \mathbb{R}^2$. The eigenvalues of $DT(x_0)$ are

$$\lambda_1 = -ax + \sqrt{a^2x^2+b} \quad \text{and} \quad \lambda_2 = -ax - \sqrt{a^2x^2+b}$$

and are real for $a^2x^2 + b \geq 0$.

Let us classify the fixed points acoding to their stability.

| The parameters a,b | Fixed Points of T |
|---|---|
| a <$-\frac{1}{4}(1-b)^2$ | None |
| $-\frac{1}{4}(1-b)^2 < a < \frac{3}{4}(1-b)^2$ | Two fixed points: one is attractor one is saddle |
| $\frac{3}{4}(1-b)^2 < a$ | Two fixed points and two attracting 2-period points |

### 5.1.2 Stable and Unstable manifolds

In this section we will demonstrate Hénon map has analytic one-dimensional stable and unstable manifolds at the saddle point $P_0$, as an application of the parameterization method for maps. From now on, we consider $a = 1.4, b = 0.3$. Since $a < \frac{3}{4}(1-b)^2$, the existence of $P_0$ is hold

Let $P_0 = (\xi_0, \eta_0)$ and $\lambda_1, \lambda_2$ the eigevalues of the derivative of T at $P_0$. Consider $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Since all the hypothesis of Theorem 1.1 for the map T are hold,

(i) $A = DT(P_0)$ is invertible since $\det A = -b \neq 0$.

(ii) $0 < \lambda_2 < 1$ and $\lambda_1$ to the invers $T^{-1}$.

(iii) $\lambda^n \notin Spec(A) = \{\lambda_1, \lambda_2\}$ for $n \geq 2$.

Then, there exists analytic maps $K : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that

$$T(K(s)) = K(\lambda s) \tag{5.2}$$

where K can be the parameterization of stable mainfold or unstable. If we denote $K(s) = (\xi(s), \eta(s))$ and put it in 5.2, we get

$$\eta(s) + 1 - a\xi^2(s) = \xi(\lambda s), \quad b\xi(s) = \eta(\lambda(s))$$

and replace the second in the first

$$\eta(s) + 1 - \frac{a}{b^2}\eta^2(\lambda s) = \frac{1}{b}\eta(s) \tag{5.3}$$

Since $K$ are analytic

$$\eta(s) = \eta_0 + \sum_{k=1}^{\infty} c_k s^k, \quad \xi(s) = \frac{1}{b}\eta_0 + \frac{1}{b}\sum_{k=1}^{\infty} c_k \lambda^k s^k \tag{5.4}$$

replacing (5.4) in (5.3) we get

$$n \geq 1 \qquad c_n = \frac{a}{b^2}\left(2\eta_0 c_n + \sum_{k=1}^{n-1} c_k c_{n-k}\right)\lambda^n + \frac{1}{b}\lambda^{2n} c_n \tag{5.5}$$

Note that for $n = 1$

$$\lambda^2 + \frac{2a}{b}\eta_0\lambda - b = 0 \tag{5.6}$$

where is the equation for eigenvalues.The choice of $c_1$ is arbitrary and fixes the scale of the parameter t, for instance we choose $c_1^{(i)} = 1$. Solving (5.5) with a change of variable $t = \lambda^n$ and the eqaution (5.6). the equation (5.5) gives

$$\begin{aligned} n > 1, \quad c_n &= \beta_n \sum_{k=1}^{n-1} c_k c_{n-k} \\ \text{where} \quad \beta_n &= \frac{-a}{b\lambda^n(1-\lambda_1\lambda^{-n})(1-\lambda_2\lambda^{-n})} \end{aligned} \tag{5.7}$$

On the one hand, the radius of convergence $\rho \neq 0$ due to $\lambda_1, \lambda_2 \neq 0$. On the other hand , if $|s|, |\lambda s|, |\lambda^2 s| < \rho$, $\eta(s)$ satisfies (5.3).

### 5.1.3 Implementation

In this section we will explain the implementation we did in order to sketch the stable and unstable manifolds. We use C programming language that can be found in annex and Gnuplot so as to draw. To draw the manifolds parameterized by $K_1$ and $K_2$, we have computed the series (5.4) truncated at the term $N = 32$ for s chosen in particular way.

#### Fundamental Domain

Firstly, we implemented `componentsManifold` so as to get the coefficients of $\eta_i$( is $\eta$ in (5.7) of $\lambda_i$) which are obtained by translating 5.7 in C. Then, we want to find a fundamental domain in this case an inteval $(-\delta, \delta)$ where $|T(K_i(s)) - K_i(\lambda_i s)| < 10^{-10}$ for each point in the interval that means the approximation is pretty good.

Let us see the correlation between N and the fundamental domain (-$\delta$, $\delta$) for fiexd point

$x = 6.313545e - 01, y = 1.894063e - 01$

| N | $\delta_1$ (Instable) | $\delta_2$ (Stable) |
|---|---|---|
| 6 | 5.85937e-03 | 2.080078e-01 |
| 8 | 1.855469e-02 | 9.990234e-01 |
| 16 | 1.357422e-01 | 9.990234e-01 |
| 32 | 5.019531e-01 | 9.990234e-01 |
| 64 | 9.990234e-01 | 9.990234e-01 |
| 128 | 9.990234e-01 | 9.990234e-01 |

Throughout the table, we can observe that the higher N is the bigger $\delta$ will be. In particular, $\delta_2$ reaches the best number faster than $\delta_1$.

We implemented `functionalEqErr` returns the error $|T(K_i(s)) - K_i(\lambda_i)|$ in order to obtain a fundamental domain. Then, we implemented `fundamDomain` as input we give the coefficients of the parameterization $K_i$ which is truncated, the eigenvalues $\lambda_i$ and an indicator (-1 or 1). $\widehat{\text{f}}$undamDomain returns

$$\begin{cases} \delta \text{ (means } [0,\delta]) & \text{if indicator} = 1 \\ -\delta \text{ (means } [-\delta,0]) & \text{if indicator} = -1 \end{cases}$$

We call the program `fundamDomain` twice with indicator = -1 and 1 for each $K_i$, we get $\delta_1$ and $\delta_2$. Then we pick the minimum and call $\delta$ that forms the fundamental domain $(-\delta, \delta)$.

### Global

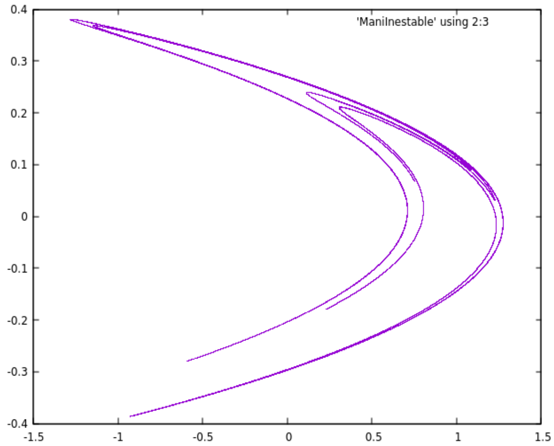
Given a fundamental domain $(-\delta, \delta)$ in which $K$ is well-defined, we extend it to all $\mathbb{R}$ by

$$K_i(s) = T^n\left(K\left(\frac{s}{\alpha_i^n}\right)\right) \qquad \text{for some n such that } \frac{s}{\lambda_i^n} \in (-\delta, \delta)$$

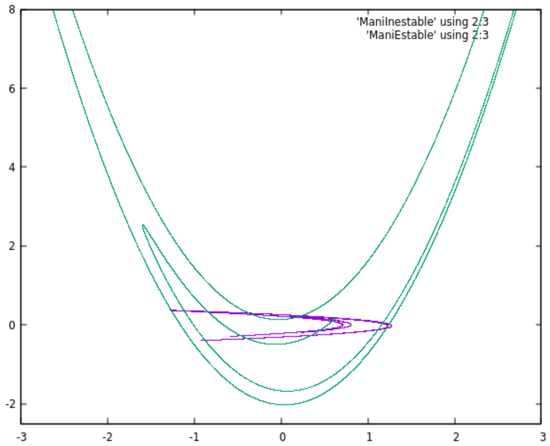
We compute $K(s)$ for points $s \in \mathbb{R}$ as follows. Assuming we have computed $K(s_0)$ we wan to compute $K(s_1)$ with $s_1 = s_0 + \Delta s_0$ so that $d_{min} < \|K(s_0) - K(s_1)\|_2 < d_{max}$ .More specifically, given $\Delta s$, $s_1 = s_0 + \Delta s$ if $s_1$ do not hold the condition, supposing it is great that $d_{max}$ we divide $\Delta s$ by 2. On the other hand,if it is less than $d_{min}$ we multiply $\Delta s$ by 2.

In our case, we consider N = 32 and the fixed point $x = 6.313545e - 01, y = 1.894063e - 01$. The following pictures are drawn by our implementation.

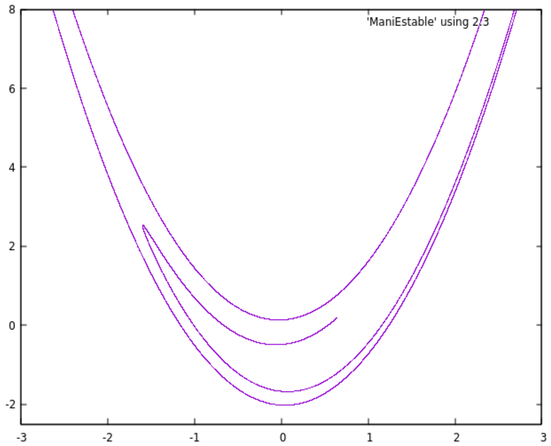
There is an interesting fact from the previous pictures. As we can see the unstable manifold is pretty similar to the Hénon Attractor.From the following They are almost same when we drawn those together.It is due to the fact that the closure of the unstable manifold contains the Hénon attractor.



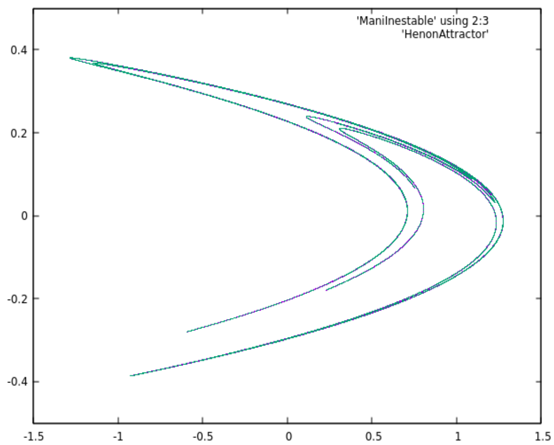

(a) Unstable Manifold




(b) Unstable Manifold and Stable Manifold




(c) Stable Manifold




(d) Unstable Manifold and Hénon Attractor

## 5.2 Lorenz System

In 1963, Edward Lorenz [Lor63] developed a simplified mathematical model for atmospheric convection. In the end he was able to obtain a system of three ordinary differential equations. Nowadays, the model is known as the Lorenz system, and it is

$$
\begin{aligned}
\frac{dx}{dt} &= a(y-x) \\
\frac{dy}{dt} &= x(b-z)-y \\
\frac{dz}{dt} &= xy-cz
\end{aligned}
\tag{5.8}
$$

where the parameters are $a$ (Prandtl number), $b$ (proportional to Rayleigh number) and $c$ (a geometric factor describing the layer of the atmosphere). In his numerical computations, Lorenz discovered the existence of chaotic solutions for certain values of the parameters ($a = 10$, $b = 28$ and $c = 8/3$), included in an attractor that is fractal in nature (a strange attractor). These facts were not rigorously proved until 2002, by Warwick Tucker [Tuc02].

Apart from the so-called Lorenz attractor, the dynamics is also dominated by the 2D stable manifold of the origin (which is a saddle point). The so-called Lorenz manifold has also a very complex structure and has been object of computational studies by many authors [KOD+05], as a test example for algorithms of computation and globalization of invariant manifolds.

The main goal of this section is to compute a the 2D stable manifold of Lorenz system and Lorenz attractor. Let us first study the fixed points, then applying the Theorem 1.5.

### 5.2.1 Stability of fixed points

The fixed points depend on the parameters. In particular, if $bleq1$ there is only a fixed point $P_0 = (0,0,0)$ and if $b > 1$ there are moreover $P_+ = \left(\sqrt{c(b-1)}, \sqrt{c(b-1)}, b-1\right)$, $P_- = \left(-\sqrt{c(b-1)}, -\sqrt{c(b-1)}, b-1\right)$.

Let us work out the case fixed point is (0,0,0) where we do implementation on. Calculating the Jacobian matrix at origin

$$A = \begin{pmatrix} -a & a & 0 \\ b & -1 & 0 \\ 0 & 0 & -c \end{pmatrix}$$

The eigenvalues(and the correpoding eigenvectors) of matrix A are

$$\begin{aligned}
\lambda_1 &= \frac{-(a+1) - \sqrt{(a+1)^2 + 4a(b-1)}}{2} = -22.828.. & v_1 &= \left(\frac{a}{a-\lambda_1}, 1, 0\right) \\
\lambda_2 &= -c = -2.667.. & v_2 &= (0,0,1) \\
\lambda_3 &= \frac{-(a+1) + \sqrt{(a+1)^2 + 4a(b-1)}}{2} = 11.828.. & v_2 &= \left(\frac{1+\lambda_3}{b}, 1, 0\right)
\end{aligned}$$

where $v_i$ stands for the eigenvector associate to eigenvalue $\lambda_i$ for $i = 1,2,3$

The following table represnts the stabiblity of the fixed points

| The parameters b | Stability of fixed points |
|---|---|
| $(-\infty, 0]$ | (0,0,0) is an attracting equilibrium |
| [1.00, 13.93] | $P_+$ and $P_-$ are attracting equilibria; the origin is unstable |
| [13.93, 24.06] | Transient chaos: There are chaotic orbits but no chaotic attractors |
| [24.06, 24.74] | A chaotic attractor coexists with attracting equilibria $P_+$ and $P_-$ |
| [24.74, ?] | Chaos: Chaotic attractor exists but $P_+$ and $P_-$ are no longer attracting |

copy from an introduction to dynamical systems PAG365

### 5.2.2 Stable and Unstable Manifolds

The idea is to use the Theorem 1.5 in order to identify the stable and ustable manifolds attached to the origin.We will consider here the computation of the 2D stable manifold belongs to Lorenz system.

Consider E the subspace generated by $v_1, v_2$ of the eigenvalues $\lambda_1, \lambda_2$ respectively. Immediately the hypothesis (i) and (ii) are satisfied due tothe fact that $Spec(A_C) = \{\lambda_3\}$ and $\lambda_3$ is positive meanwhile the elements of $Spec(A_E)$ are negative. We can consider L = 1, that is, $R$

can be chosen to be a linear vactor field we have to see there are no interval resonances between eigenvalues $\lambda_1 < \lambda_2 < 0$.

Interval resonances correspond to couples $(j_1, j_2) \in \mathbb{N} \times \mathbb{N}$ such that $j_1 + j_2 \geq 2$ and

$$\begin{aligned} \text{either} \quad & j_1\lambda_1 + j_2\lambda_2 = \lambda_1 \\ \text{or} \quad & j_1\lambda_1 + j_2\lambda_2 = \lambda_2 \end{aligned}$$

The second case is not possibles, since $\lambda_1 < \lambda_2 < 0$. The first one would be possible only if $j_1 = 0$ and then $j_2 = \frac{\lambda_1}{\lambda_2} \in \mathbb{N},\ \ j_2 \geq 2$. In our case, $\frac{\lambda_1}{\lambda_2} \simeq 8.559...$ That is not an integer. Hence, the are not interval resonance and $R$ can be chosen to be a linear vector field.

The last hypothesis consists in the elements of $Spec(-A) + 2Spec(A_E)$ whose real part is negative. Since the hipothesis (ii) hold, it is sufficient to check with $Spec(-A_E) + 2Spec(A_E)$. Immediately, we can see $Spec(-A_E) + 2Spec(A_E) \subset \{z \in \mathbb{C} \mid Rez < 0\}$.

The existence of 2D stable manifold at origen $W^s(P_0)$ is straightforward application of the Theorem 1.5. Therefore, there exists a parameterization $K : U_1 \subset E \to \mathbb{R}^3$ and $R : \mathbb{R}^2 \to \mathbb{R}^2$ of $W^s(P_0)$

$$K(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} \quad s \in \mathbb{R}^2, \quad R(s) = \Lambda s \text{ where } \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfy the functional equation $\mathcal{X} \circ K = DK \cdot R$ in $U_1$ Since $W$ is analytic, we write

$$K(s) = \sum_{k=0}^{\infty} K_k(s) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} K_{\ell,k-\ell} s_1^{\ell} s_2^{k-\ell} = \begin{pmatrix} \sum_{k=0}^{\infty} x_k(s) \\ \sum_{k=0}^{\infty} y_k(s) \\ \sum_{k=0}^{\infty} z_k(s) \end{pmatrix} \tag{5.9}$$

Using the proceding notations in $\mathcal{X} \circ K = DK \cdot R$, we get

$$\sum_{k=0}^{\infty} a(y_k(s) - x_k(s)) = \lambda_1 \frac{\partial x}{\partial s_1} s_1 + \lambda_2 \frac{\partial x}{\partial s_2} s_2 \tag{5.10}$$

$$\sum_{k=0}^{\infty} \left( b x_k(s) - y_k(s) - \sum_{j=1}^{k-1} x_j(s) z_{k-j}(s) \right) = \lambda_1 \frac{\partial y}{\partial s_1} s_1 + \lambda_2 \frac{\partial y}{\partial s_2} s_2 \tag{5.11}$$

$$\sum_{k=0}^{\infty} \left( \sum_{j=1}^{k-1} x_j(s) y_{k-j}(s) - c z_k(s) \right) = \lambda_1 \frac{\partial z}{\partial s_1} s_1 + \lambda_2 \frac{\partial z}{\partial s_2} s_2 \tag{5.12}$$

Fix the order k of the right-hand of 5.10

$$\begin{aligned} \lambda_1 \frac{\partial x_k}{\partial s_1} s_1 + \lambda_2 \frac{\partial x_k}{\partial s_2} s_2 &= \lambda_1 \cdot \sum_{\ell=0}^{k} \ell \cdot x_{\ell,k-\ell} s_1^{\ell} s_2^{k-\ell} + \lambda_2 \cdot \sum_{\ell=0}^{k} (k-\ell) \cdot x_{\ell,k-\ell} s_1^{\ell} s_2^{k-\ell} \\ &= \sum_{l=0}^{k} (\lambda_1 \ell + \lambda_2 (k-\ell)) \cdot x_{l,k-l} s_1^{\ell} s_2^{k-\ell} \end{aligned} \tag{5.13}$$

Using the same idea to the right- hand of 5.11, 5.12. By matching the power $\ell, k-\ell$, we get the coefficients of the term $s_1^{\ell} s_2^{k-\ell}$

$$a \cdot y_{\ell,k-\ell} - a \cdot x_{\ell,k-\ell} - (\lambda_1 \ell + \lambda_2 (k-\ell)) x_{\ell,k-\ell} = 0 \tag{5.14}$$

$$b \cdot a_{\ell,k-\ell} - y_{\ell,k-\ell} - (\lambda_1 \ell + \lambda_2 (k-\ell)) y_{\ell,k-\ell} = +(xz)_{\ell,k-\ell} \tag{5.15}$$

$$-c z_{\ell,k-\ell} - (\lambda_1 \ell + \lambda_2 (k-\ell)) z_{\ell,k-\ell} = -(xy)_{\ell,k-\ell} \tag{5.16}$$

where $(xz)_{\ell,k-\ell}$ stands for the coefficient of term $s_1^\ell s_2^{k-\ell}$ of $\sum_{j=1}^{k-1} x_j(s)z_{k-j}(s)$, $(xy)_{\ell,k-\ell}$ stands for the coefficient of term $s_1^\ell s_2^{k-\ell}$ of $\sum_{j=1}^{k-1} x_j(s)y_{k-j}(s)$ which the two summations are known.

We obtain a $3 \times 3$ system of equations to the coefficients of $x, y, z$. In fact, we can solve (5.16) and consider a $2 \times 2$ system of equation for instance use Cramer rule to solve it. Note that for k = 1 the equations (5.14), (5.15) and (5.16) are equal to the equations for eigenvalues to $\lambda_1, \lambda_2$, that is, $W_{0,1} = v_1$ and $W_{1,0} = v_2$.

Roughly speaking, the computation of 1D unstable follows from analogous way to 2D.

### 5.2.3 Reduce Dynamics

As the Theorem stated R is a representation of the dynamics of the vector field $\mathcal{X}$ restricted to the manifold. In the present case, $R(s) = \Delta s$ and the reduce a vector field is

$$\begin{aligned} \dot{u} &= \lambda_1 u \\ \dot{v} &= \lambda_2 v \end{aligned}$$

we can solve the above equations separetely, getting the solutions

$$\begin{aligned} u(t) &= e^{\lambda_1 t} u_0 \\ v(t) &= e^{\lambda_2 t} v_0 \end{aligned} \tag{5.17}$$

Moreover, evaluating in the parameterization $K$ we will know the dynamics of the manifold.

### 5.2.4 Numerical integration of solutions: Taylors method

Consider the Lorenz system (5.8) with initial condition $x(t_0) = x_0, y(t_0) = y_0$ and $z(t_0) = z_0$. Using Taylor's fomula, the following equations such as x(t) is the approximation of $x(t_0 + h)$ where $h = t - t_0$

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} x_k (t - t_0)^k = \sum_{k=0}^{\infty} x_k h^k \\ y(t) &= \sum_{k=0}^{\infty} y_k (t - t_0)^k = \sum_{k=0}^{\infty} y_k h^k \\ z(t) &= \sum_{k=0}^{\infty} z_k (t - t_0)^k = \sum_{k=0}^{\infty} z_k h^k \end{aligned} \tag{5.18}$$

We put the preceding notations in left and right-hand of Lorenz system 5.8

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k=1}^{\infty} x_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) x_{k+1} t^k \\ a(y - x) &= a \cdot \sum_{k=0}^{\infty} (y_k - x_k) t^k \end{aligned}$$

by matching the powers and following the preceding way to $y$ and $z$, we will get

$$x_{k+1} = \frac{a(y_k - x_k)}{k+1}$$
$$y_{k+1} = \frac{b \cdot x_k - y_k - \sum_{j=0}^{k} x_j z_{k-j}}{k+1}$$
$$z_{k+1} = \frac{\sum_{j=0}^{k} x_j y_{k-j} - c z_k}{k+1}$$

This is the basis of a numerical integrator. If we want the error in one step will be less than a tolerance such as $\epsilon = 1e^{-16}$, that is,
$|x_{N+1}|\, h^{N+1} < \epsilon$

$$x(t) = \sum_{k=0}^{N} x_k h^k + x_{N+1} h^{N+1} + x_{N+2} t^{N+2} + \cdots$$

Hence, $h_{N+1} < \sqrt[N+1]{\frac{\epsilon}{\|(x_{N+1}, y_{N+1}, z_{N+1})\|}}$. In fact we calculate $H_{N+2}, H_{N+3}$ and choose the h = $\min(H_{N+1}, H_{N+2}, H_{N+3})$. This is an advantage over other methods, since the local error is round-off error and the control step is based on a direct computations of $h$.

### 5.2.5 Implementation

#### 2D stable Manifold

In this subsection we will show the implementation we did in order to draw the 2D stable manifold whose form looks like a butterfly demonstrated in figure 5.6b. The code, which are programmed in C language, will be found in annex. Furthermore we use Gnuplot in order to draw the graphics. To draw the 2D stable manifold parameterized , we have computed the series 5.9 truncated at the term $N = 25$.

We programmed `lorenz_manifold.c` consists in working out the coefficients of the parameterization $K$ then give points to the program `Horner2D` and write down the result in a file named `Manifold` where `Horner2D` will return the value of evaluating $(s_1, s_2)$ in the $W^{\leq N}$

Firstly, we want to explain how to store each component of the $W$. We will declare an array named $Wx$ whose $k$-th component will point to the another array of length k. That is, $Wx[k][\ell]$ stands for the coefficient of the term $s_1^{\ell} s_2^{k-\ell}$of $x_k$, which $x_k$ is a homogeneous polynomial of degree k. Analogous to the component $y, z$ of $W$.

Secondly, we implemented the Cramer rule in order to get the $x_{\ell,k-\ell}, y_{\ell,k-\ell}$ in (5.14), (5.15) and $z_{\ell,k-\ell}$ is easy got from the equation (5.16). Once we have the coefficients of $W^{\leq}$ evaluating points in W we can draw the Lorenz maniold. We did a *Horner2D*, which is extended the Horner method to two variables, to evaluate points $(s_1, s_2)$ in a finite summation of homogeneous polynomials.

Finally, the way to choose the points we neither evaluate in random points nor find a fundamental domain as case of Hénon map. Instead of that, we gave the points belong to a ball of radius $r = 20$ center at the origin $B_r(0)$ and wrote down the results and the error committed of trucanting the serie. To obatin the error we programmed `errFunctionalEq` as output will return the error $\|\mathcal{X}(K(s)) - DK(s)\Lambda s\|$. Note that evaluating the results obtained in `Horner2D` given $s = (s_1, s_2)$ at the Lorenz system we get the first term in the norm. We did `Horner2D_1` to obtain the second term and the explicit formula for the sencod term showed up in (5.13). We called it `Horner2D_1` due to use a similar strategy to `Horner2D`.

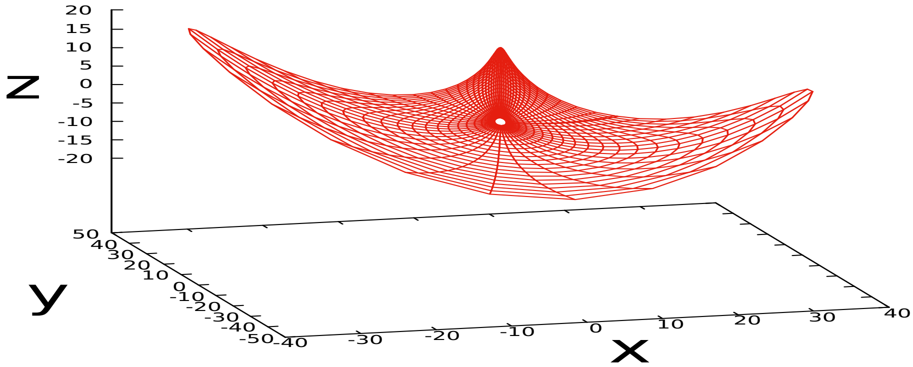
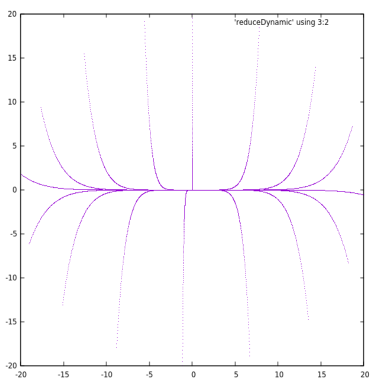


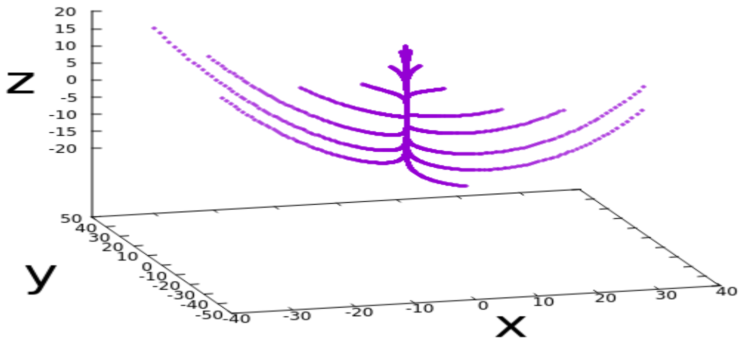

Figure 5.3: 2D Stable Manifold of Lorenz system

### Reduce dynamics

In this subsection, we want to see the 2D reduce dynamic and 3D in graphics. Using the provious knowledge of reduce dynamic. First, we get the points $(u, v)$ applying the equation (5.17). Then evaluate in each component of the parameterazation with using the `Horner2d`. Finally we write down the result in a file that we can draw it.




(a) 2D reduce dynamic

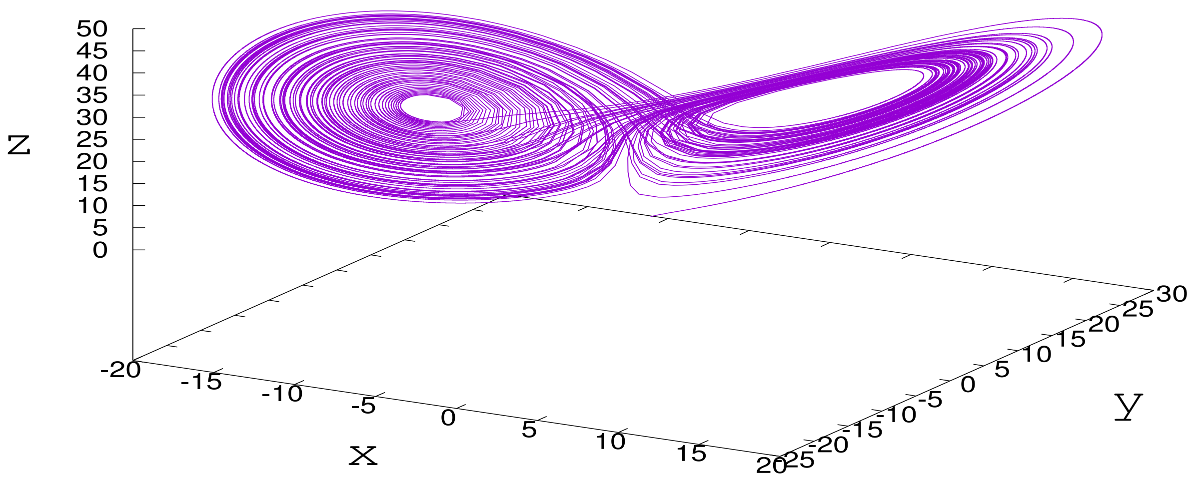



(b) 3D reduce dynamic

**Lorenz Attractor**

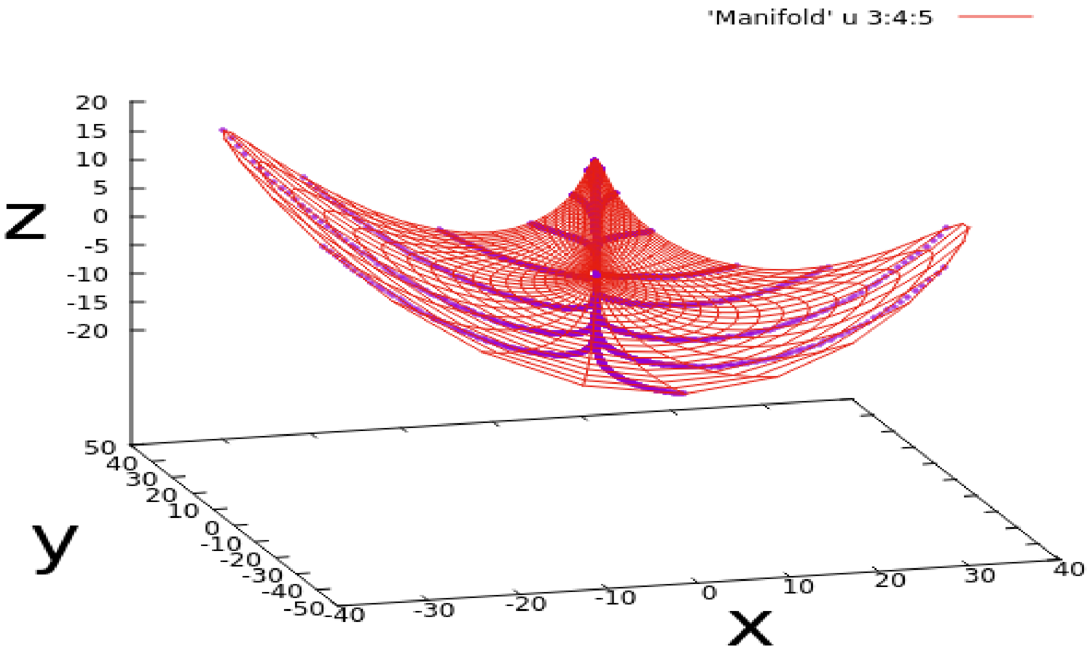
In this section we will explain the method called integration we used to draw the Lorenz attractor. Since we have calculated the coefficient of x(t), which x(t) is the approximation of $x(t_0 + h)$. To draw the Lorenz attractor we have computed the series truncated at N= 25.

We programmed `integration.c` consists in giving an initial condition and evaluating in the truncated series in recursive way, that is, given an initial condition $(x_0, y_0, z_0)$ and evaluate in the series we obtain $(x_1, y_1, z_1)$ and write down in a file called `integration`. Then, reapeat the process, but use initial condition $(x_1, y_1, z_1)$ due to $x(t), y(t), z(t)$ are the approximation of $x(t_0 + h), y(t_0 + h), z(t_0 + h)$. In each step h we chose is provided that the error is smaller than $\epsilon$. Do this process successively while $t < 100$, we are able to draw the Lorenz attractor.

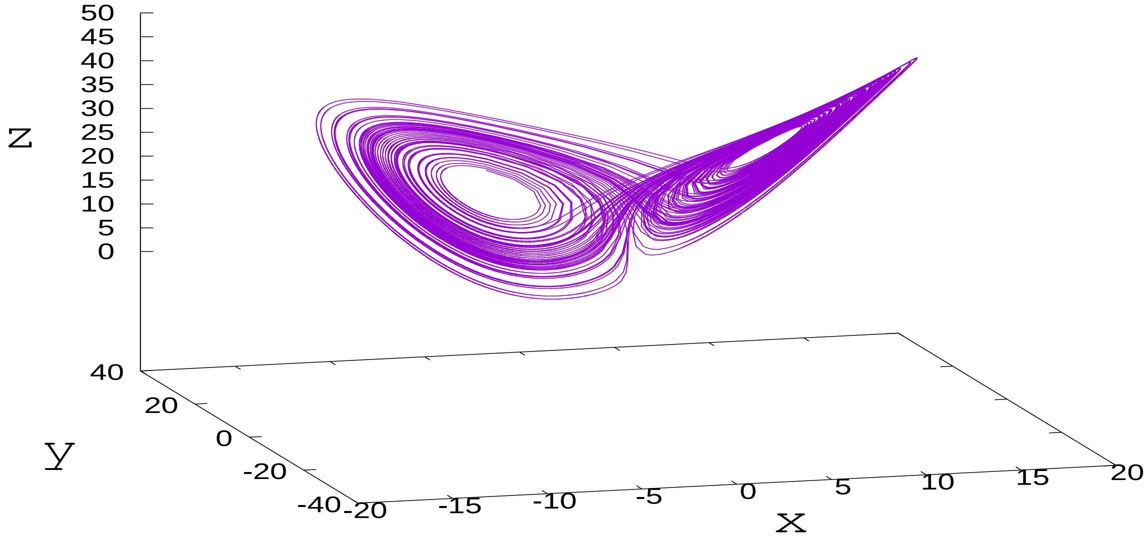



(a) Lorenz Attractor

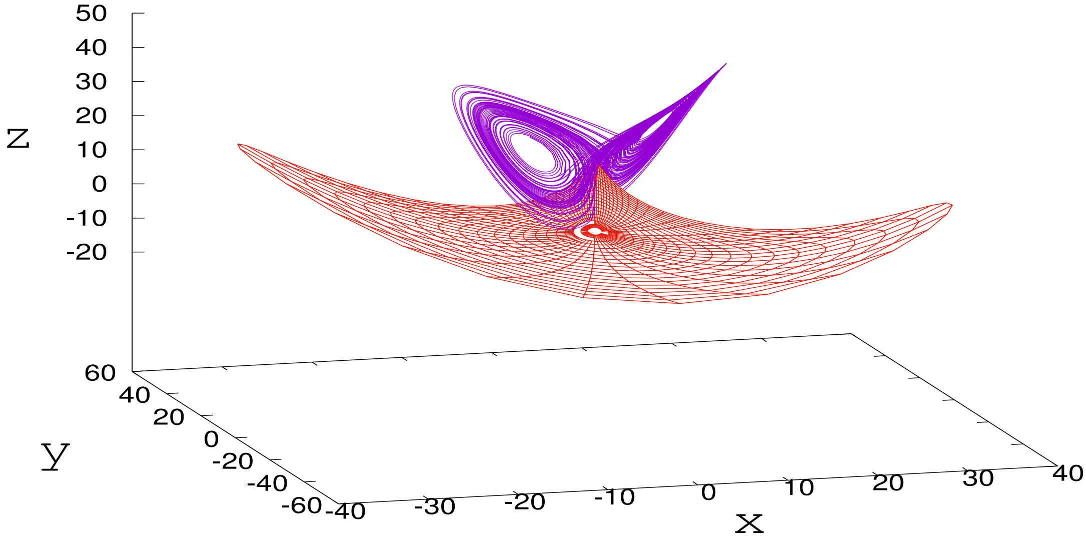
Bla, bla [Sim79, CFdlL03a], [CFdlL03b], [HCF+16]



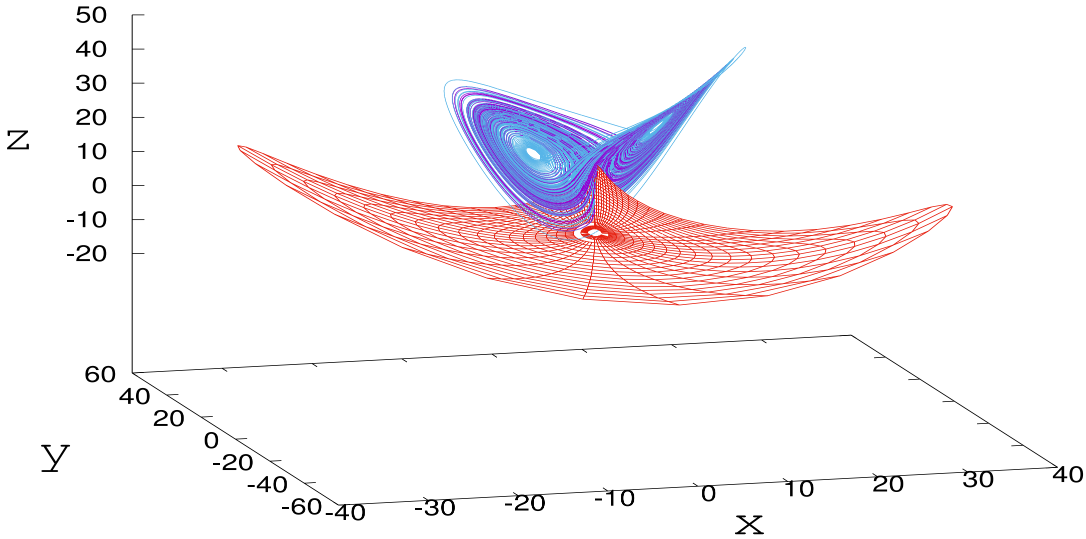

(a) Manifold and its dynamincs




(b) Lorenz Attractor




(a) Stable Manifold, Lorenz Atractor




(b) Unstable Manifold, Stable Manifold and attractor

# Conclusion

In this work we have developed the parameterization method for invariant manifolds of real-analytic dynamic systems. We have seen how powerful it is, as a method to study the existence and the regularity of these objects in diverse contexts, as well as the development of effective computational algorithms.

Due to the complex work that it is, we just see a part of the parameterization method(real-analytic). In fact, we left some work to do such as globalize the fundamental domain in Lorenz system, see how the automatic differential is.

# Annex I

# Hénon map

To compute the (un)stable manifold and attrator of the Hénon map we use the following codes.

```
#include<math.h>
#include<stdio.h>
#include<stdlib.h>

#define error 1.e-10
#define a 1.4
#define b 0.3
#define N 32
#define pi acos(-1.0)


int componentsManifold(double *, double, double);
double functionalEqErr(double *,double, double);
double fundamDomain(double *, double, int);
void Manifold(double *C, double s0, double alpha, double delta, int indicator, int);
void Fn(double z[2] ,double zn[2], int );
void evaluate(double zn[2],double s, double *C, double alpha, double delta);
double hornerEvaluation(double *C,double s0);

int main(void){
  /*main idea: In this program we try to sketch the Henon attractor, Estable Manifold and Inestable Manifold of fixed point(we have two fixed point depen on the a and b could be all saddle-node or one atractor and another slow atractor)
  componentsManifold: give the coefficient of the expansion of the estable manifold or inestable manifold.
  functionalEqErr: return the error of functional equation |F(Wn(s)-wn(lambda s)|
  fundamDomain: return the fundamental domain [0,deltaf](indicator 1) or [-deltab,0](indicator = -1)
  nextpoint: give s0 nextpoint return s1 such that dmin<|w(s1)-w(s0)| < dmax
  Fn: return the n-th evaluation of F at point x
  evaluate: return the Fn(w(s/(lambda)^n))
  */
  int i,choice, count;
  double alpha1, alpha2, fixPoint1x, fixPoint2x,fixedPointX;
```

```
    double dmin, dmax,delta1f,delta1b,delta2f, delta2b, delta1, delta2;
    double point, point1f, point2f,point1b, point2b, s,apprX, apprY;
    double *eta1, *eta2, z1[2],z0[2];
    double dif;

    /*alpha1 alpha2 are eigenvalues of fixed point that |alpha1|>1 and |alpha2
        |<1*/
    /*gamma1, gamma2 are the second component of the expansion of unestable and
        estable manifold respectly */

    dmin = 1.e-5;
    dmax = 1.e-1;
    point1f = 0;
    point1b = 0;
    point2f = 0;
    point2b = 0;
    count = 0;
    delta1 = 0;
    delta2 = 0;

    eta1 = (double *)malloc(N*sizeof(double));
    eta2 = (double *)malloc(N*sizeof(double));

    if (eta1 == NULL || eta2 == NULL){
    printf("problem of memory \n");
    exit;
    }

    FILE *file1;
    FILE *file2;
    FILE *file3;

    file2 = fopen("ExpansionManiEstableY","w");
    file1 = fopen("ExpansionManiInestableY", "w");
    file3 = fopen("HenonAttractor","w");

    if(file1 == NULL || file2 == NULL || file3 == NULL){
      printf("problem in opening output files\n");
    }

    /*fixPoint1x is the first component of fixedPoint and x = fixPoint1x y=0.3*x
        */
    fixPoint1x = (b-1 + sqrt((b-1)*(b-1)+4*a))/(2*a);
    fixPoint2x = (b-1 - sqrt((b-1)*(b-1)+4*a))/(2*a);

    printf("we have two fixed points, which one do you prefer?\n");
    printf("the first fixed point %le %le\n", fixPoint1x,0.3*fixPoint1x);
    printf("the second fixed point %le %le\n", fixPoint2x,0.3*fixPoint2x);
    printf("enter 1 for fixed point 1 and enter 2 for fixed point 2\n");
    scanf("%d",&choice);

    if(choice == 1){
      fixedPointX = fixPoint1x;
    }else{
      fixedPointX = fixPoint2x;
    }
```

```

    printf("to sketch the  Henon Attractor we need a point near Fixed Point, so enter the difference for each component (difX, difY)\n");
    printf("For instance, 0.001 for x \n 0.001 for y\n");
    scanf("%le", &apprX);
    scanf("%le", &apprY);


    alpha2 = -a*fixedPointX + sqrt((2*a*fixedPointX)*(2*a*fixedPointX) + 4*b)/2;
    alpha1 = -a*fixedPointX - sqrt((2*a*fixedPointX)*(2*a*fixedPointX) + 4*b)/2;
    printf("alpha1 %le\n alpha2 %le\n",alpha1,alpha2);
    /*the fisrt component of each eta is the second component of fixedPoint*/
    eta1[0] = 0.3*fixedPointX;
    eta2[0] = 0.3*fixedPointX;

  componentsManifold(eta2, alpha1, alpha2);
  componentsManifold(eta1, alpha2, alpha1);

    //we are going to write separaly the each component of estable and inestable manifold in diferent files.
    for(i = 0; i < N; i++){
      fprintf(file1,"%d %le\n",i,eta1[i]);
      fprintf(file2,"%d %le\n",i,eta2[i]);
    }

    // Using fundamDomain we obtain two deltals and we choose the minimum between
    delta1f = fundamDomain(eta1,alpha1,1);
    delta1b = fundamDomain(eta1,alpha1,-1);
    if (fabs(delta1b) < delta1f){
      delta1 = fabs(delta1b);
    }else{
      delta1 = delta1f;
    }

    //Same way to alpha2
    delta2f = fundamDomain(eta2,alpha2,1);
    delta2b = fundamDomain(eta2,alpha2,-1);
    if (fabs(delta2b) < delta2f){
      delta2 = fabs(delta2b);
    }else{
      delta2 = delta2f;
    }

    /*since we have the fundamental domain [-delta1, delta1] and [delta2, delta2] for ManiInestable and ManiEstable*/
    //checking
    /*for (s= 0; s< delta1*pow(fabs(alpha1),5); s+= 0.001){
      evaluate(z,s,eta1,alpha1,delta1);
      fprintf(file4,"%le %le %le \n",s, z[0], z[1], functionalEqErr(eta1,alpha1,s));
    }
    for (s= 0; fabs(s)< delta1*pow(fabs(alpha1),5); s-= 0.001){
      evaluate(z,s,eta1,alpha1,delta1);
      fprintf(file4,"%le %le %le %le \n",s, z[0], z[1], functionalEqErr(eta1, alpha1, s));
```

```
    }*/

    /*Henon's Attractor*/
    //first we do a transient such as 10000 steps
    z0[0] = fixedPointX + apprX;
    z0[1] = 0.3*fixedPointX + apprY;
    Fn(z0, z1, 10000);
    z0[0] = z1[0];
    z0[1] = z1[1];

    for(i = 0; i < 10000; i++){
    Fn(z0, z1, 1);
    fprintf(file3,"%le %le\n", z1[0],z1[1]);
    z0[0] = z1[0];
    z0[1] = z1[1];
    }

    //print the points of Manfolds
    Manifold(eta1, 0, alpha1, delta1, 1, -1);
    Manifold(eta1, 0, alpha1, delta1, -1, -1);
    Manifold(eta2, 0, alpha2, delta2, 1, 1);
    Manifold(eta2, 0, alpha2, delta2, -1, 1);


    fclose(file1);
    fclose(file2);
    fclose(file3);
    free(eta1);
    free(eta2);
    return 0;
    }

  int componentsManifold(double *C2,double alpha12, double alpha21){
    /*This function return the coffiecient of the expansion of the manifolds using a explicit formulate to calculate these.
    C is where we are going to save the coefficient and
    The first component of expansion is the the sencond compnent of the fixed point that que chose.
    The second compnent of expansion is arbitrary, we choose 1

    alpha12 and alpha21 are eigenvalues of fixed point that we chose and there are 2 cases:
    case1:
    alpha12 = alpha1 and alpha21 = alpha2  this function returns the coefficient for Estable Manifold.
    case2:
    alpha12 = alpha2 and alpha12 = alpha1 this function returns the coefficient for Inestable Manifold.
    */
    int n,j,k;
    double beta, an;

    C2[1] = 1;
    an= alpha21;

    for(n = 2; n < N; n++){
```

```
      an*= alpha21;
      beta = -a/(b*an*(1-alpha12/an)*(1-alpha21/an));
      C2[n] = 0;
      for(k = 1; k < n; k++){
        C2[n] +=C2[k]*C2[n-k];
    }
      C2[n] = C2[n]*beta;
    }
    return 0;

  }

  double fundamDomain(double *C,double alpha, int indicator){
    /*this function return a fundamental domain where the error of functional
        equation is small that an  epsilon that is given

    C is the expansion of Manifold could be the estable one or the inestable.
    alpha corresponds to the eigenvalue of estable or inestable Manifold.
    indicator could be 1 or -1
    case indicator = 1
    return [0,delta]
    case indicator = -1
    return [-delta ,0]
    */
    int  count;
    double I1, I2, pn, delta, err;

    delta = 0;
    count = 0;
    I1 = 0;
    I2 = indicator*1;
    pn = (I1 + I2) / 2 ;

    do{
      if((err= functionalEqErr(C,alpha,pn)) < error){
      delta = pn;
      I1 = pn;
      pn = (I1 + I2)/2;
      /*if(indicator ){
        printf("count %d pn %le err %le\n",count,pn, err);
      }*/
      }else{
      I2 = pn;
      pn = (I1 + I2) /2;
      /*if(indicator ){
        printf("count %d pn %le err %le\n",count,pn, err);
      }*/
    }

    count++;
    }while(count<10);
    return delta;
    }

  double functionalEqErr(double *C, double alpha, double t){
    /*This funciton returns the error of the functional equation at point t
```

```
  the parameters C, alpha are the same as before
  */
  double sol, aux1, aux2, aux3;
  int i;

  aux1 = 0;
  aux2 = 0;
  aux3 = 0;

  aux1 = hornerEvaluation(C,t);
  aux2 = hornerEvaluation(C,alpha*t);
  aux3 = hornerEvaluation(C,alpha*alpha*t);

  sol = fabs(aux1 +1 - a*aux2*aux2/(b*b)-aux3/b);
  return sol;

}

void Manifold(double *C, double s0, double alpha, double delta, int indicator, int TypeMani){
  /*This function returns the estable Manifold(TypeMani = 1)  or Inestable Manifold(TypeMani = -1) depen on the parameter TyperMani
  given C,alpha,delta as before
  where indicator the points obtained are foward or backward
  */
  int i;
  double  z0[2], z1[2], d, dmin, dmax, longitud, dif;

  dmin = 1.e-4;
  dmax = 1.e-2;
  longitud = 0;
  dif = 0.001;
  FILE *file;



  if (TypeMani == 1){
    file = fopen("ManiEstable","w");
  }else{
    file = fopen("ManiInestable","w");
  }

  if(file == NULL  ){
    printf("problem in opening output files\n");
  }

  do{
    evaluate(z0, s0, C, alpha, delta); //z0 = ws0
    evaluate(z1, s0 + indicator*dif, C, alpha, delta); //z1 = ws1
    d = hypot(z0[0]-z1[0], z0[1]-z1[1]); //hypotenuse = distance between z0 and z1

    if (d > dmax){
      dif /=  2;
    }else{
      fprintf(file,"%le %le %le \n",s0, z1[0], z1[1]);
```

```
        s0 += indicator*dif;
        longitud += d;
      if(d < dmin){
        dif *= 2;
      }
      }
    }while(longitud < 1000);
    //we stop the do while  when th length of the curve is great than 1000
    fclose(file);
    return ;
  }


  void Fn(double z[2] ,double zn[2], int n){
    /*this funtion returns n-th evaluation of Henon Map at z=(z[0],z[1]) in a
        vector call zn = (zn[0], zn[1])
    Wath out!
    if n > 0 we evaluate in F and if n < 0 we evaluate in inverse of F and n=0
        we don't evaluate retu the same*/
    double x,y;
    int i;

    x= z[0];
    y = z[1];
    if (n > 0){
      for(i = 1; i <= n; i++){
        zn[0] = y +1 -a*x*x;
        zn[1] = b*x;
        x= zn[0];
        y = zn[1];
      }
    }else if(n == 0){
      zn[0] = x;
      zn[1] = y;

    }else{
      for(i = -1; i >= n; i--){
        zn[0] = y/b;
        zn[1] = x-1+a*y*y/(b*b);
        x = zn[0];
        y = zn[1];
      }
    }

    return;
  }

  void evaluate(double zn[2],double s, double *C, double alpha, double delta){
    /*This funciton returns w(s) where we work out and obtain w(s) = Fn(w(s/
        lambda^n))
    watch out !!
    if fabs(alpha) < 1 we need to multiply instead of divide*/

    int i;
    double aux,sn,n,x,y,z[2];
    if (fabs(s) <= delta){
```

```
      n = 0;
    }else{
    if (fabs(alpha) > 1){
    //this is the formulate obtained after working out the condition such that fabs(s/lambda^n) < delta
      n = ceil( log(fabs(s)/delta) / log(fabs(alpha)));
    }else{
      n = floor( log(fabs(s)/delta) / log(fabs(alpha)));
      }
    }
    s = s / pow(alpha, n);
    // evaluation in component x
    y = hornerEvaluation(C,s);
    //evaluation in component Y
    x = hornerEvaluation(C,alpha*s);
    x = x/b;
    z[0] = x;
    z[1] = y;

    Fn(z,zn,n);
  }

  double hornerEvaluation(double *C,double s0){
    double bn,bn_1;
    int i;

    bn = C[N];

    for(i = N; i > 0; i--){

    bn_1 = C[i-1]  + bn*s0;
    bn = bn_1;

    }
  return bn;
  }
```

# Annex I

# Lorenz system

To compute the 2D stable manifold we have programmed the following codes

```
#include<math.h>
#include<stdio.h>
#include<stdlib.h>

#define error 1.e-10
#define tolerancia 1.e-16
#define NW 25
#define pi acos(-1.0)
double a = 10.0;
double b = 28.0;
double c = 8.0/3.0;

double Horner1D(double *C, double x, int N);
double Horner2D(double **W, double u, double v, int N);
double errFunctionalEq(double **Wx,double **Wy,double **Wz,double *alpha,
    double u,double v,int N);
double Horner2D_1(double **W, double u, double v, double *alpha,int N);

int main(void){
  /*LORENZ SYSTEM
  dx/dt = a(y-x)
  dy/dt = bx -xz -y
  dz/dt = xy-cz
  Main idea: to sketch the twe-dimensional stable manifold
  Horner2D: evaluate u,v in a homegeneous polynomial up to term N.
  */
  double **Wx, **Wy, **Wz;
  double alpha[3], M[3][3]= { {-a,a,0},{b,-1,0},{0,0,-c} }, P[3][3];
  int i, j, k, l, i1, i2, j1, j2, k0;
  double xz, xy, A, B, C, D, E, F, norm, err;
  double wx,wy, wz,du,dv,ut,vt;
  double check, u, v, r,t, th, maxerr;

  FILE *file1;
  FILE *file2;
  FILE *file3;
  file1 = fopen("Manifold","w");
  file2 = fopen("reduceDynamic","w");
```

```
    file3 = fopen("Dynamic","w");
    if(file1 == NULL || file2 == NULL){
      printf("there a problems in opening file\n ");
      exit (-1);
    }

    /*we save the component x,y,z of the satble manifold in W, Wy, Wz respectively.
    where Wx[k] is the vector for saving the coeffifient of the homogeneous polynomial of degree k and Wx[k][l] the coefficient of the term u^l v^(k-l) of the homogeneous polinomial of degree k.Same to Wy,Wz.
    */
    Wx= (double **) malloc((NW+3)*sizeof(double*));
    Wy= (double **) malloc((NW+3)*sizeof(double*));
    Wz= (double **) malloc((NW+3)*sizeof(double*));

    for(k = 0; k<= NW+2; k++){
    Wx[k]= (double*) malloc((k+1)*sizeof(double));
    Wy[k]= (double*) malloc((k+1)*sizeof(double));
    Wz[k]= (double*) malloc((k+1)*sizeof(double));

    if (!Wx[k] || !Wy[k] || !Wz[k]) {
      puts("malloc problems");
      exit(-1);
      }
    }

    alpha[0] = -((a+1)/2) - (sqrt((a+1)*(a+1) + 4*a*(b-1)))/2;
    //printf("alpha 0: %lf\n", alpha[0]);
    P[0][0] = 1;
    P[1][0]= (alpha[0]-M[0][0])/M[0][1];
    P[2][0] = 0;

    alpha[1] = -c;
    //printf("alpha 1: %lf\n", alpha[1]);
    P[0][1]= 0;
    P[1][1]= 0;
    P[2][1]= 1;

    alpha[2] = -((a+1)/2) + (sqrt((a+1)*(a+1) + 4*a*(b-1)))/2;
    P[0][2] = 1;
    P[1][2]= (alpha[2]-M[0][0])/M[0][1];
    P[2][2] = 0;

    //normalize the eigenvector
    for (j= 0; j<3; j++){
      norm= 0;
      for (i= 0; i<3; i++) {
        norm+= P[i][j]*P[i][j];
    }
      norm= sqrt(norm);
      for (i= 0; i<3; i++) {
        P[i][j]/= norm;
      }
    }

```

```
    // order 0 : the fix point
    Wx[0][0]= 0.;
    Wy[0][0]= 0.;
    Wz[0][0]= 0.;

    // order 1: the first column of P is the eigenvector for alpha1, the second
        for alpha2
    Wx[1][0]= P[0][1]; Wx[1][1]= P[0][0];
    Wy[1][0]= P[1][1]; Wy[1][1]= P[1][0];
    Wz[1][0]= P[2][1]; Wz[1][1]= P[2][0];

    for (k= 2; k<= NW+2; k++){
      for (l= 0; l<= k; l++) {
      // Equation for W[k][l]

      // Computation of xy, xz
        xy= xz= 0;
        for (i1= 0; i1<= k; i1++) {
          for (j1= 0; j1<= l; j1++) {
          i2= k-i1;
          j2= l-j1;

          if (j1<= i1 && j2<= i2) {
            xy+= Wx[i1][j1]*Wy[i2][j2];
            xz+= Wx[i1][j1]*Wz[i2][j2];
            }
          }
        }

        Wz[k][l]= xy/(c+alpha[0]*l+alpha[1]*(k-l));
        //Using the Cramer rule to solve the linaer system
        A= -a-alpha[0]*l-alpha[1]*(k-l); B= a;
        C= b; D= -1-alpha[0]*l-alpha[1]*(k-l);
        E= 0; F= xz;

        Wx[k][l]= -F*B/(A*D-B*C);
        Wy[k][l]=  F*A/(A*D-B*C);

        }
    }

  //compute the 2D stable manifold, given points from balls center at the origin
       with radius r 1<= r <=20

  for (r= 1; r<= 20; r+= 1){
    maxerr= 0;
    for (th= 0; th<=2*M_PI+0.05; th+= 0.1) {
      u = r*cos(th);
      v = r*sin(th);
      wx = Horner2D(Wx,u,v,NW+2);
      wy = Horner2D(Wy,u,v,NW+2);
      wz = Horner2D(Wz,u,v,NW+2);

      fprintf(file1,"% le % le % le % le % le % le\n", u, v,
      Horner2D(Wx,u,v,NW+2),Horner2D(Wy,u,v,NW+2),Horner2D(Wz,u,v,NW+2),
      err= errFunctionalEq(Wx,Wy,Wz,alpha,u,v,NW+2));
```

```
      if (err>maxerr) maxerr= err; //maxerr stands for the maximum error in each
           ball
      }
      fprintf(file1,"\n#% le % le\n", r, maxerr);
      }

  //Reduce Dynamic, we give the points in the closure of the ball with radius 20
    for(th = 0; th <=2*M_PI+0.05; th+= 0.4){
      u = 20*cos(th);
      v = 20*sin(th);
      t = 0;
      fprintf(file2,"\n#% le \n",th);
      do{
        ut = exp(alpha[0]*t)*u;
        vt = exp(alpha[1]*t)*v;
        t += 0.001;
        //reduce dynamic 2D
        fprintf(file2,"% le % le %le\n",t,ut,vt);
        //the dynamic of the system
        fprintf(file3,"% le % le % le % le % le % le\n", u, v,
        Horner2D(Wx,ut,vt,NW+2),Horner2D(Wy,ut,vt,NW+2),Horner2D(Wz,ut,vt,NW+2),
        errFunctionalEq(Wx,Wy,Wz,alpha,ut,vt,NW+2));
      }while(fabs(ut)> 0.01  || fabs(vt)> 0.01);

    }
    return 0;
    fclose(file1);
    fclose(file2);
    fclose(file3);
  }

  double Horner1D(double *C, double x, int N){
    double bn;
    int i;

    for(i= N, bn= C[N]; i > 0; i--)
    bn = C[i-1] + bn*x;

    return bn;
    }

  double Horner2D_1(double **W, double u, double v, double *alpha,int N)
  {
    // Horner2D_1 returns DW(s)·R(s) where we reformulate(see memory) and the
        programa will return the truncated series evaluting in DW(s)·R(s) by
        using a kind of horner method
    double w= 0, C;
    int l1, l2;

    w= alpha[1]*N*W[N][0];

    for (l2= N; l2>0; l2--) {
      // C[l2-1]
      C= (alpha[0]*(N-l2+1)+ (l2-1)*alpha[1])*W[N][N-l2+1];
      for (l1= N-l2+1; l1>0; l1--){
        C= (alpha[0]*(l1-1)+ alpha[1]*(l2-1))*W[l2+l1-2][l1-1] + C*u;
```

```
      }

      w= C + w*v;
    }

  return w;
  }


  double Horner2D(double **W, double u, double v, int N)
  {
    //Horner2D returns F(K(s)) by using a kind of horner method in 2 variable
    double w= 0, C;
    int l1, l2;

    w= W[N][0];

    for (l2= N; l2>0; l2--) {
      // C[l2-1]
      C= W[N][N-l2+1];
      for (l1= N-l2+1; l1>0; l1--){
        C= W[l2+l1-2][l1-1] + C*u;
      }

    w= C + w*v;
    }

  return w;
  }

  double errFunctionalEq(double **Wx,double **Wy,double **Wz,double *alpha,
      double u,double v,int N){
    //this program will return the modul of error of functional equation by
        using Horner2D and Horner2D_1.
    double wx,wy,wz,dwx,dwy,dwz,errx,erry,errz,errFunctionalEQ;

    wx = Horner2D(Wx,u,v,N);
    wy = Horner2D(Wy,u,v,N);
    wz = Horner2D(Wz,u,v,N);

    dwx = Horner2D_1(Wx,u,v,alpha,N);
    dwy = Horner2D_1(Wy,u,v,alpha,N);
    dwz = Horner2D_1(Wz,u,v,alpha,N);

    errx = pow(fabs(a*(wy-wx) - dwx),2);
    erry = pow(fabs(wx*(b-wz)-wy - dwy),2);
    errz = pow(fabs(wx*wy-c*wz - dwz),2);
    errFunctionalEQ = sqrt(errx + erry + errz);

    return errFunctionalEQ;
  }
```

and to compute the Lorenz attractor and the unstable manifold we have programmed the following codes

```
#include<math.h>
#include<stdio.h>
```

```
#include<stdlib.h>

#define error 1.e-10
#define tolerancia 1.e-16
#define N 25
#define pi acos(-1.0)
double a = 10.0;
double b = 28.0;
double c = 8.0/3.0;

void componentSolution(double *x, double *y, double *z);
double hornerEvaluation(double *C,double s0);
void integration(double *x, double *y,double *z,int indicator);

int main(void){
  /*LORENZ SYSTEM
  dx/dt = a(y-x)
  dy/dt = bx -xz -y
  dz/dt = xy-cz
  Main idea: Using the calculation that we have done, we are going to compute and sketch the Lorenz attractor.
  componentSolution: return the solution for initial condition (x0,y0,z0)
  hornerEvaluation: evaluate a point in a polynomial.
  integration: using the method explained in memory. Given a initial condition s0 we calculate a approximation the solution of Lorenz system in s1 = s0 +h.
  */
  double x[N+3], y[N+3], z[N+3],alpha, alpha1;

  FILE *file1;
  file1 = fopen("componentSolution","w");
  if(file1 == NULL){
    printf("there a problems in opening file\n");
    exit(-1);
  }
  //eigenvalue
  alpha = -((a+1)/2) + (sqrt((a+1)*(a+1) + 4*a*(b-1)))/2;
  alpha1 = -((a+1)/2) - (sqrt((a+1)*(a+1) + 4*a*(b-1)))/2;
  //eigenvector
  double v1[3] = {1, (alpha +a)/a, 0};
  double v2[3] = {1, (alpha1 +a)/a, 0};

  // printf("eigenvector =  % le % le %le \n eigenvalues = % le ",v[0],v[1],v[2],alpha);
  printf("enter the initial condition for the system: ");
  scanf("%lf %lf %lf", x, y, z);
  printf("initial conditions are %le %le %le\n",x[0],y[0],z[0]);
  integration(x,y,z,2); //Given the initial condition by using integration we will able to get the point in attractor

  //Given a point that is a eigenvector1 multiply by 0.1, we using the intragration draw the orbital in positive time.
  x[0] = 0.1*v1[0];
  y[0] = 0.1*v1[1];
  z[0] = 0.1*v1[2];
  integration(x,y,z,1);
```

```

  //Given a point that is a eigenvector2 multiply by -0.1, we using the intragration draw the orbital in positive time.
  x[0] = -0.1*v2[0];
  y[0] = -0.1*v2[1];
  z[0] = -0.1*v2[2];
  integration(x,y,z,-1);

  return 0;

}

double hornerEvaluation(double *C, double x){
double bn;
  int i;

  for(i= N, bn= C[N]; i > 0; i--)
  bn = C[i-1] + bn*x;

return bn;
}
void integration(double *x, double *y,double *z,int indicator){
//interation will be able to calculate Lorenz attractor(when indicator = 2) and Unstable manifold(indicator = -1 for eigenvector2 or indicator = -1 eigenvector2).
  double x0, y0, z0, dx, dy, dz,n;
  double h1, h2, h, t, tmax, dmax =0.1,dim = 0.0001;
  int bool;
  FILE *file1;

  if (indicator == 2){
    file1 = fopen("componentSolution","w");
    tmax= 200;
  }else if(indicator == 1){
    file1 = fopen("UnstableManifold1","w");
    tmax= 100;
  }else {
    file1 = fopen("UnstableManifold2","w");
    tmax= 100;
  }
  if(file1 == NULL){
    printf("there a problems in opening file\n");
    exit(-1);
  }


  for (t= 0; fabs(t)< tmax; ){

    componentSolution(x, y, z);
    // where h1 satnds for the error with coefficient of the n+1-th term that we commit in each step and h2 with n+2.
    h1 = pow(tolerancia/sqrt(x[N+1]*x[N+1] + y[N+1]*y[N+1] + z[N+1]*z[N+1]), 1./(N+1));
    h2 = pow(tolerancia/sqrt(x[N+2]*x[N+2] + y[N+2]*y[N+2] + z[N+2]*z[N+2]), 1./(N+2));
    //take the minimum
```

```
    h= (h1 < h2) ? h1 : h2;


    //(x0,y0,z0) stands the flow at the point (x0,y0,z0) after occuring t time .
    x0= x[0];
    y0= y[0];
    z0= z[0];

    //We vanish the the first component in each vector x,y,z in order to work out the better h.
    x[0]= 0;
    y[0]= 0;
    z[0]= 0;

    //if the the norm of the system is great that dmax we will reduce h.
    while( dx= hornerEvaluation(x, h),
      dy= hornerEvaluation(y, h),
      dz= hornerEvaluation(z, h),
      sqrt(dx*dx+dy*dy+dz+dz)>dmax ){
      h/= 2.;
    }
    while( dx= hornerEvaluation(x, h),
      dy= hornerEvaluation(y, h),
      dz= hornerEvaluation(z, h),
      sqrt(dx*dx+dy*dy+dz+dz) <dim){
      h*= 2.;
    }

    //dx is obtained evaluating in h in x with out the first component, respect to dy,dz are obtained in same way.Now, x[0],y[0],z[0] are got by evaluating at h and initial in flow condition(x0,y0,z0).In a addition it will be the new initial condition for the next round.

    x[0]= x0 + dx;
    y[0]= y0 + dy;
    z[0]= z0 + dz;
    t += h;

    if (!(indicator== 2 & fabs(t)<100)) {
      fprintf(file1, "% le % le % le % le\n", t, x[0], y[0], z[0]);
    }
  }
  fclose(file1);
  return ;
}

void componentSolution(double *x, double *y, double *z){
  //using the formula we worked out in memory
  int k, j;
  double xz,xy;

  for(k = 1; k <= N+2; k++){
    xz = 0;
    xy = 0;
    for(j = 0; j < k; j++){
```

```
        xz += x[j]*z[k-1-j];
        xy += x[j]*y[k-1-j];
      }
      x[k] = (a*(y[k-1] -x[k-1]))/k;
      y[k] = (b*x[k-1] -y[k-1] -xz)/k;
      z[k] = (xy -c*z[k-1])/k;

    }
  }
```

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